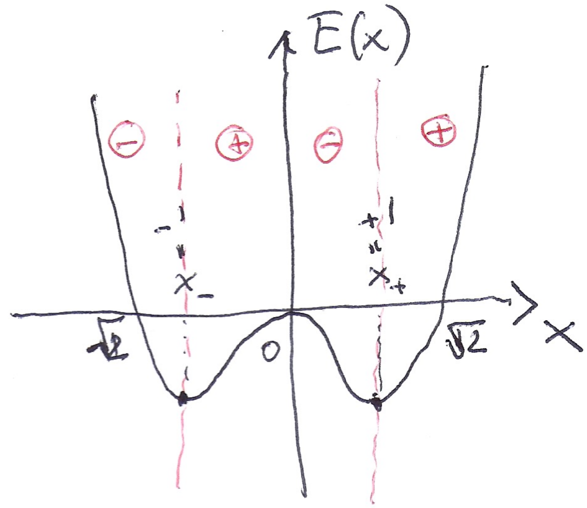
Bistable system (switch)

↳ one-dimensional energy-based model



$$E(x) = \frac{x^2}{2}\left(\frac{x^2}{2} - 1\right)$$

$$E'(x) = \frac{\partial E}{\partial x} = x^3 - x = x(x^2 - 1)$$

Downhill dynamics?

$$\begin{cases} x \text{ increasing when } E'(x) < 0 \\ x \text{ decreasing when } E'(x) > 0 \end{cases}$$

ODE: ordinary differential equation. (about $x(t)$)

$$\tau \dot{x} = \tau \frac{\partial x}{\partial t} = -E'(x) = -\frac{\partial E}{\partial x}(x), \quad x(0) = x_0$$

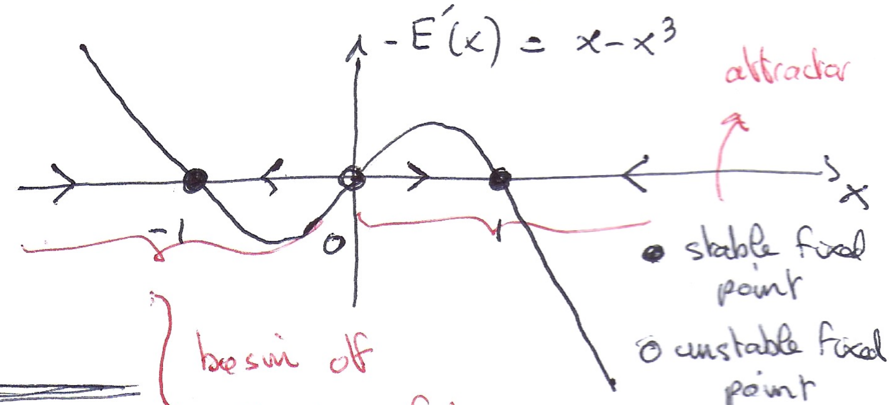
initial condition

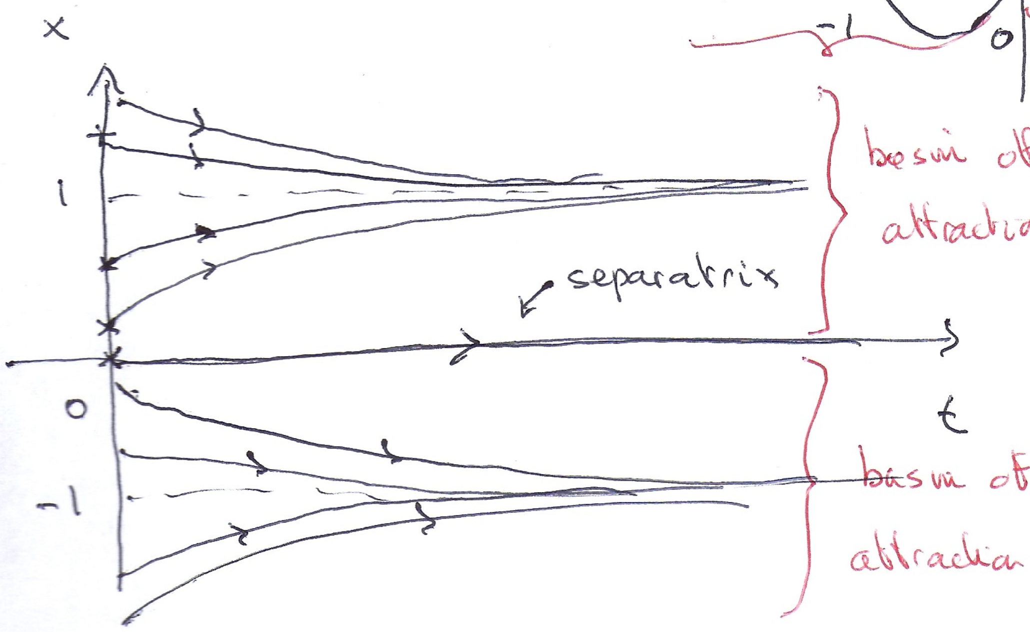
downhill? $E(x(t))$: $\frac{\partial}{\partial t} E(x(t)) = E'(x(t)) \frac{\partial x}{\partial t} = E'(x(t))\left(-\frac{E'(x(t))}{\tau}\right)$

↑ Lyapunov function

$$= -\frac{1}{\tau} E'(x(t))^2 \leq 0$$

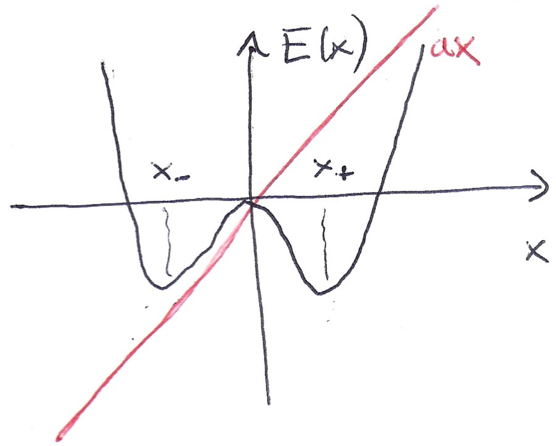
Typical behavior:





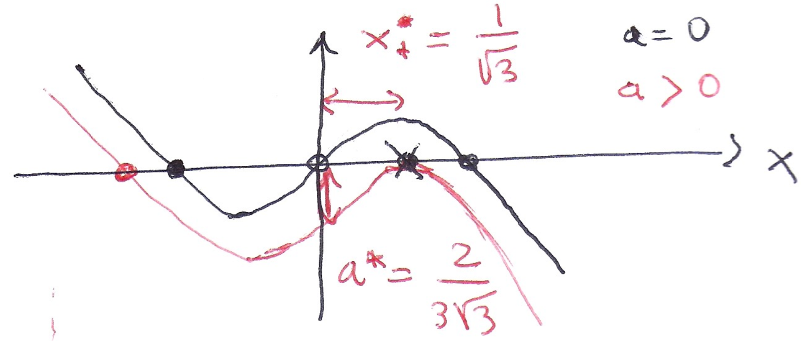
Oscillatory system (switch + adaptation)

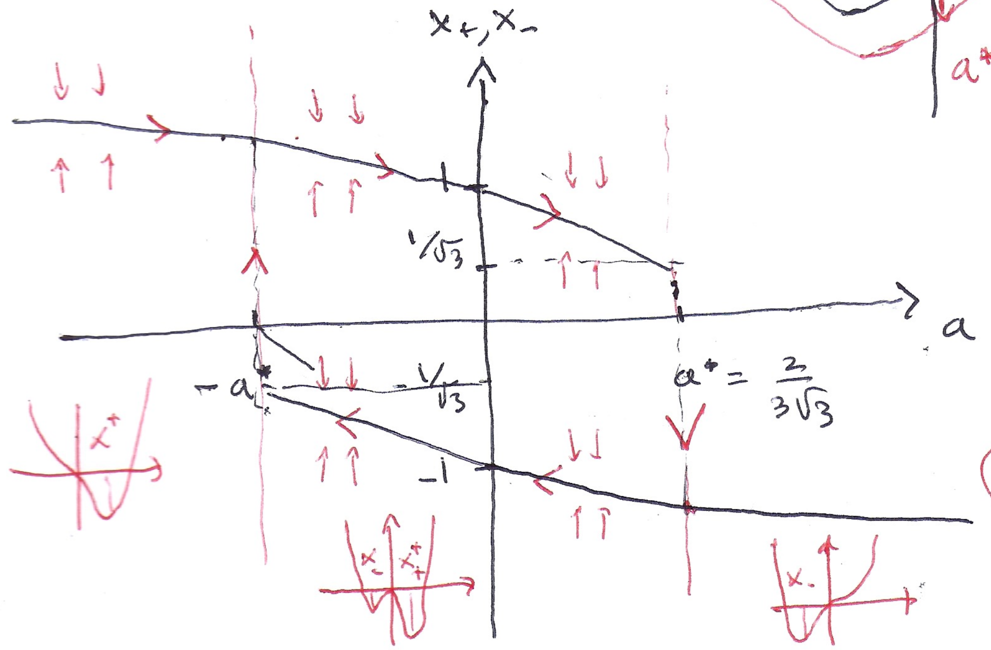
↳ idea: introducing a control parameter "a" favoring an attractor "x_+" at the expense of "x_-" (and reciprocally)



$$E_a(x) = \frac{x^2}{2}\left(\frac{x^2}{2} - 1\right) + ax$$

$$-E_a'(x) = x - x^3 - a$$





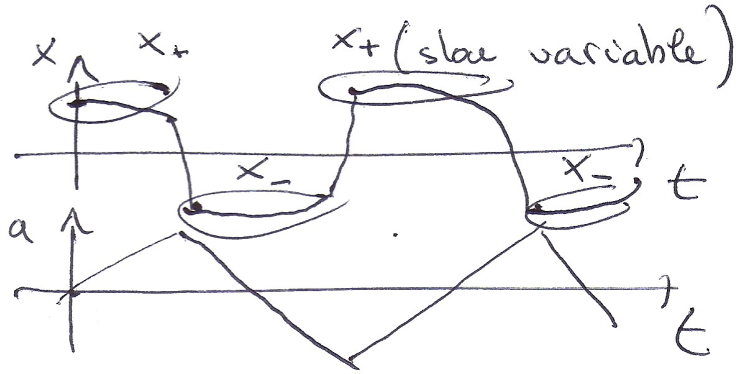
slowly varying a

$a(t) \leftarrow$ control protocol

hysteresis

↳ idea: coupled dynamics of x and a to get oscillation

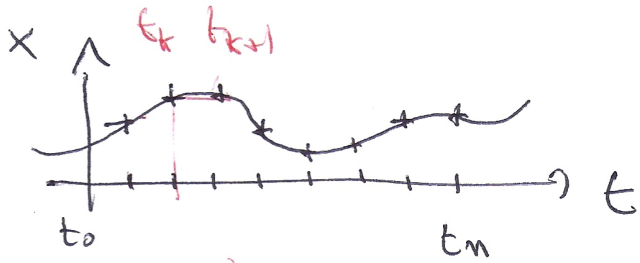
$$\begin{cases} \tau_x \dot{x} = -E_a'(x) = x - x^3 - a & \text{(fast variable)} \\ \wedge \\ \tau_a \dot{a} = x & \text{(slow variable)} \end{cases}$$



Numerical simulation (ODEs)

↳ discretization scheme : $\tau \dot{x} = x - x^3 = f(x)$

$x(t) \to$ continuous t



$t_n = n \Delta t$ ← time step

$(t_k, f(t_k)) \to \tau \dot{x}(t_k) = f(t_k) \to x(t_{k+1}) = x(t_k) + \dot{x}(t_k) \Delta t$

$\dot{x}(t_k) = \frac{f(t_k)}{\tau}$

$$\tau \dot{x} = \tau \frac{dx}{dt} = f(x)$$, discrete approximation of

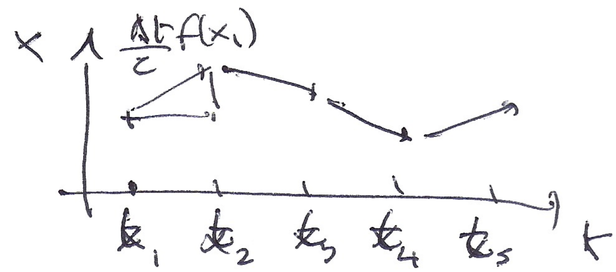
$$\frac{dx}{dt} = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} = \frac{x_{k+1} - x_k}{\Delta t}$$

↕

$$\tau \frac{x_{k+1} - x_k}{\Delta t} = f(x_k)$$: Euler forward scheme

~~$f(x_{k+1})$: backward~~

$$x_{k+1} = x_k + \frac{\Delta t}{\tau} f(x_k)$$



There are many such schemes : [accuracy, stability

$\Delta t \to 0$