Modeling by dynamical system

↳ mathematical theory: ordinary differential equation

$$\tau \dot{\underline{x}} = \underline{f}(\underline{x})$$

time constant, time derivative, modeling

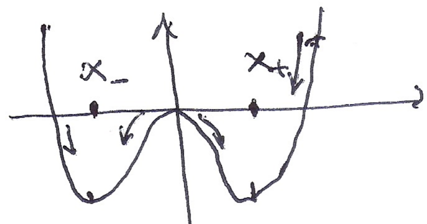
Ex: energy-based model (gradient model)

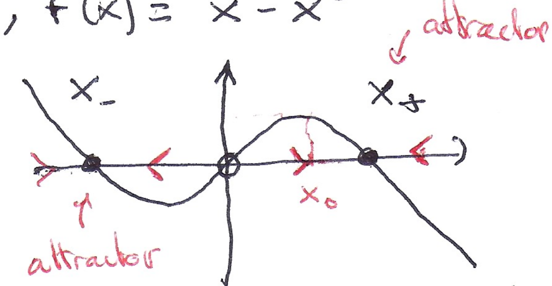
$$\underline{f}(\underline{x}) = -\nabla_{\underline{x}} E(\underline{x}) \underset{1D}{=} -\frac{\partial E}{\partial x} = -E'(x)$$

$$\nabla_{\underline{x}} E(x_1, x_2) = \left(\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2}\right)$$

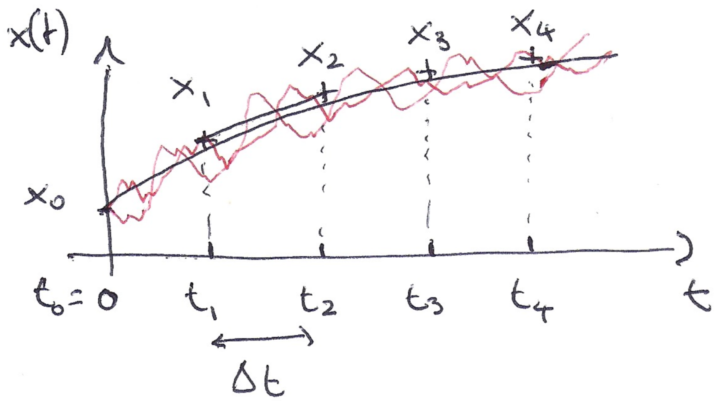
Bistable case: $E(x) = \frac{x^2}{2}\left(\frac{x^2}{2}-1\right)$, $f(x) = x - x^3$

$x(t)$





↳ numerical simulation: discretization scheme (Forward Euler)



discrete derivative

$$\tau \frac{x_{k+1} - x_k}{\Delta t} = f(x_k)$$

update rule:

$$x_{k+1} \leftarrow x_k + \frac{\Delta t}{\tau} f(x_k) + \text{noise}$$

$$\sqrt{\Delta t}\, \eta_k$$

small Δt: more precise but more costly.

Goal of today: including noise

$$\tau \dot{\underline{x}} = \tau \frac{\partial \underline{x}}{\partial t} = f(\underline{x}, \eta)$$

$\eta(t)$

noise: randomly fluctuating perturbation

"additive noise"

$$= f(\underline{x} + \eta) \quad \text{or} \quad f(\underline{x}) + \eta$$

Modeling noise in dynamical systems

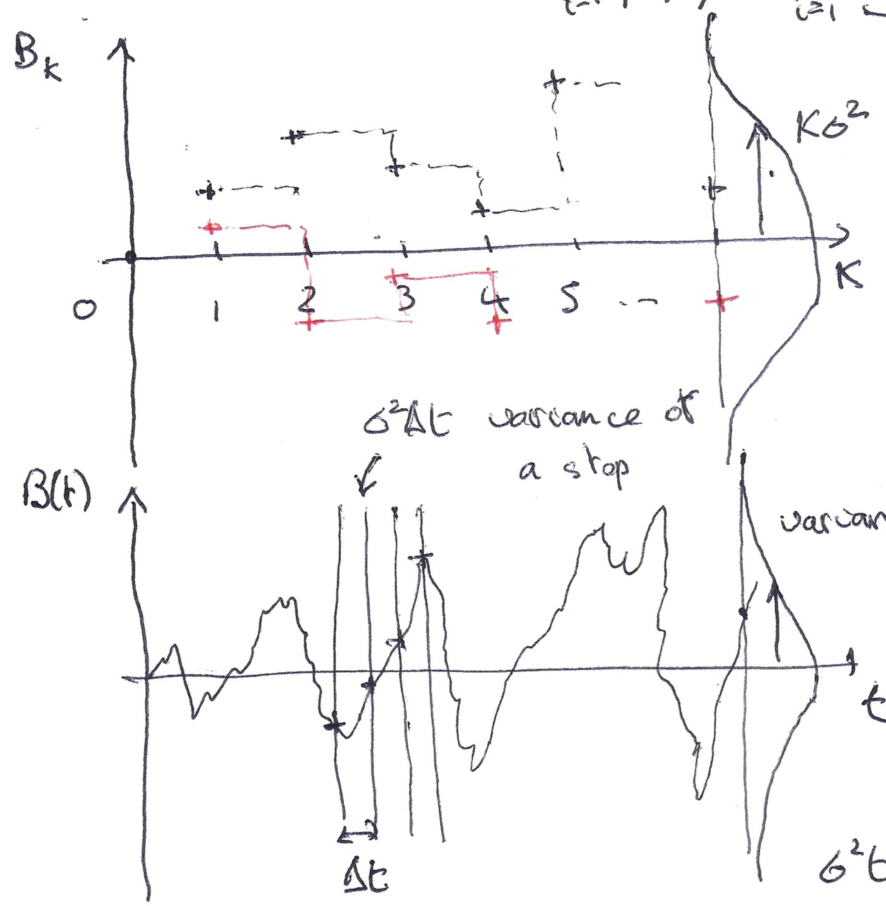
↳ Inspiration: Brownian motion

$\eta_1, \eta_2, \ldots, \eta_K$: independent Gaussian random variables

$\langle \eta_i \rangle = 0 \qquad \langle \eta_i^2 \rangle = \sigma^2$

$B_K = \eta_1 + \eta_2 + \cdots + \eta_K$: Gaussian too. $\langle \sum_{i=1}^{K} \eta_i \rangle = \sum_{i=1}^{K} \underbrace{\langle \eta_i \rangle}_{0} = 0$

$$\langle (\sum_{i=1}^{K} \eta_i)^2 \rangle = \sum_{i=1}^{K} \underbrace{\langle \eta_i^2 \rangle}_{\sigma^2} + \sum_{i \neq j} \cancel{\langle \eta_i \eta_j \rangle}_{0} = K\sigma^2$$



← random walk B_K

↑ integers

↑ discrete in time

← Brownian motion

↑ continuous in time

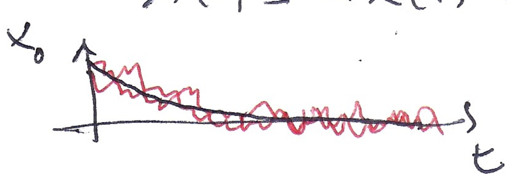
$$\sigma^2 t = \sigma^2 \left(\frac{t}{\Delta t}\right) \Delta t = \sigma^2 \text{ "}k\text{" } \Delta t$$

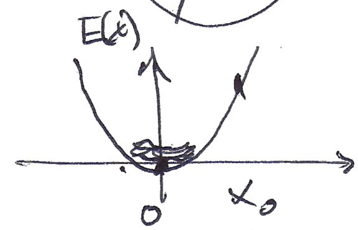
# of steps

↳ Theory: stochastic differential equation (SDE)

$B(t)$ is solution of $\dot{B}(t) = \eta(t)$ ← Gaussian white noise

$\dot{x}(t) = -x(t) + \eta(t)$:





$E(x) = \frac{x^2}{2}$

$-E'(x) = -x$

↳ Numerics

$\dot{x}(t) = \eta(t)$ ← Gaussian white noise

$x_0, x_1, \ldots, x_k$ ← discretized value

$t_0, t_1, \ldots, t_k$; $t_k = k\Delta t$

← time step

Euler:
$$\boxed{\frac{x_{k+1} - x_k}{\Delta t} = a\,\eta_k}$$

η_k ← Gaussian random variable $N(0, \sigma^2 = 1)$

$\frac{x_{k+1} - x_k}{\Delta t}$: discretized derivative

a: parameter to choose consistently

* By definition of the Brownian motion, we have (want)

$$\langle x_k^2 \rangle = \langle B(k\Delta t)^2 \rangle = \sigma^2 (k\Delta t)$$

* By definition of the Euler scheme, we have

$$x_k = x_{k-1} + a\Delta t\,\eta_{k-1} = a\Delta t \sum_{i=0}^{k-1} \eta_i$$

$$\langle x_k^2 \rangle = a^2 \Delta t^2 \left\langle \left(\sum_{i=0}^{k-1} \eta_i \right)^2 \right\rangle = a^2 \Delta t^2 k$$

To be consistent, we need: $\sigma^2 \cancel{k} \cancel{\Delta t} = a^2 \Delta t^{\cancel{2}} \cancel{k}$

$$a = \frac{\sigma}{\sqrt{\Delta t}}$$

In general:
$$\boxed{x_{k+1} = x_k + \Delta t\, f(x_k) + \sqrt{\Delta t}\,\eta_k}$$

$\sqrt{\Delta t}\,\eta_k$ ← noise