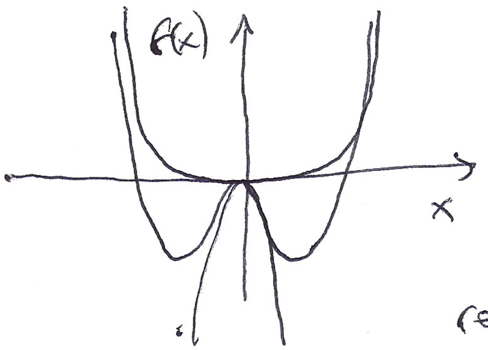
Fourier analysis

↳ Idea: function representation ($\neq$ ways to look at the same function)

Taylor series



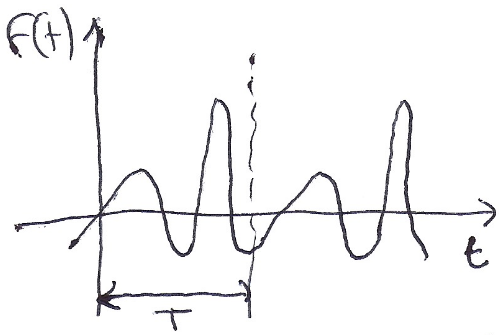
Taylor coefficient

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \dots = \sum_{n \geq 0} a_n x^n$$
$$= 0 + 0 - x^2 + 0 + x^4$$

$$\begin{array}{ccc} f(x) & \xrightarrow{f^{(n)}(0)/n!} & a_n \\ & \xleftarrow{\sum_{n\geq 0} a_n x^n} & \\ \text{regular function} & & \text{Taylor coeff} \end{array}$$

Computionally interesting but fundamentally local (hard to approximate bounded functions)

Fourier series (periodic function)



$$f(t) = a_0 + a_1 \cos\left(\frac{2\pi}{T} t\right) + b_1 \sin\left(\frac{2\pi}{T} t\right)$$
$$+ a_2 \cos\left(\frac{2\pi}{T} 2t\right) + b_2 \sin\left(\frac{2\pi}{T} 2t\right)$$
$$\vdots$$
$$+ a_k \cos\left(\frac{2\pi}{T} kt\right) + b_k \sin\left(\frac{2\pi}{T} kt\right)$$
$$\vdots$$

periodic function

$f(t)$

"time domain"

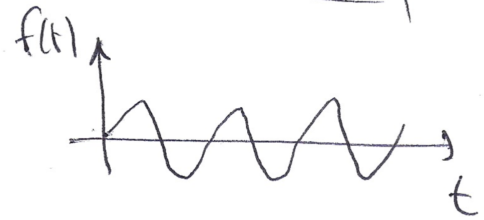
$$a_k = \frac{1}{T}\int_0^T f(t)\cos\left(\frac{2\pi}{T}kt\right)dt$$
$$b_k = \frac{1}{T}\int_0^T f(t)\sin\left(\frac{2\pi}{T}kt\right)dt$$

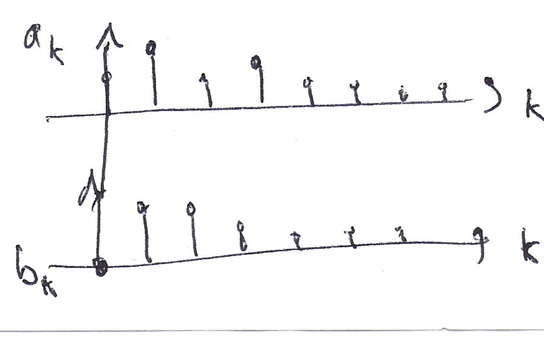
Fourier coeffs

(a_k, b_k) $\quad b_0 = 0$

"frequency domain"

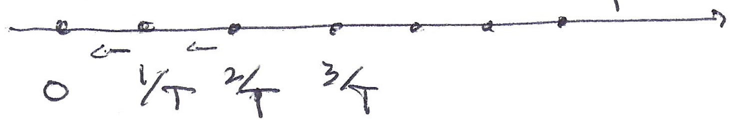
$$\sum_{k\geq 0} \left(a_k \cos\left(\frac{2\pi}{T}kt\right) \; b_k \sin\left(\frac{2\pi}{T}kt\right)\right)$$





Fourier transform (non-periodic function) $T \to +\infty$

$\omega_k = \frac{k}{T}$ $\qquad \omega_0 \quad \omega_1 \quad \omega_2 \quad \omega_3 \qquad \Delta\omega = \frac{1}{T} \to 0$



$T \to +\infty$ all possible frequencies will matter

$\omega_k \hookrightarrow \omega$ real number (even negative)

$$f(t) \xrightarrow{\int_{-\infty}^{+\infty} f(t)\, e^{-i2\pi\omega t}\, dt} \hat{f}(\omega),\ F(\omega)$$

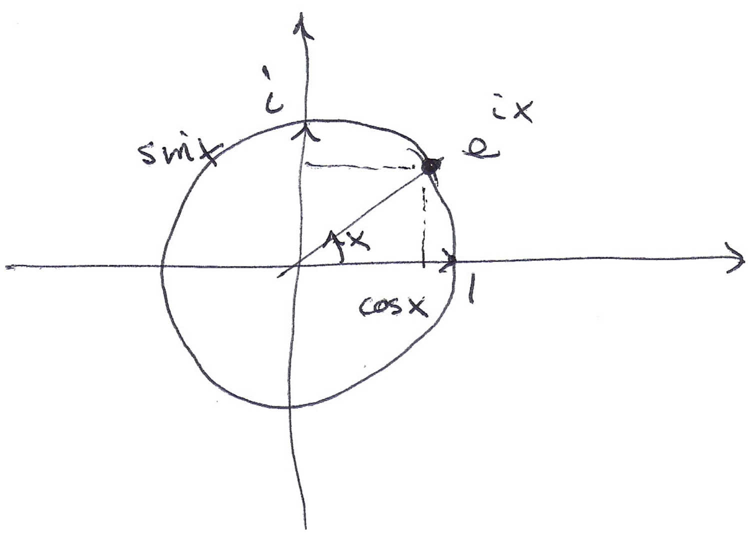
regular function "time domain"

Fourier transform "frequency domain"

$$f(t) \xleftarrow{\int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{i2\pi\omega t}\, d\omega} \hat{f}(\omega)$$

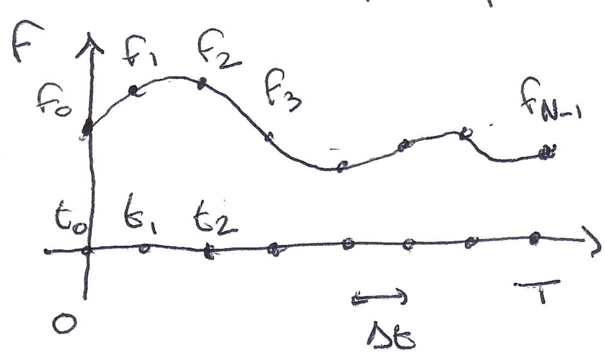
real $\qquad$ imaginary unit

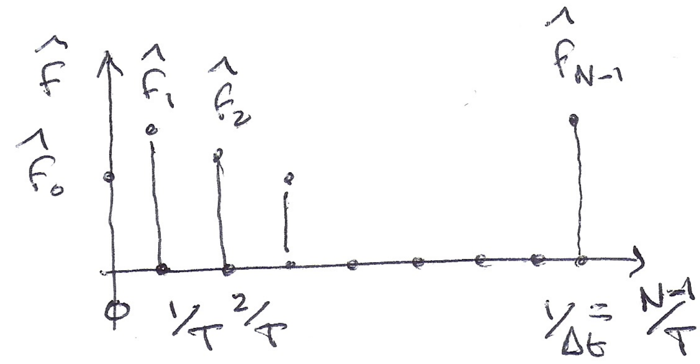
Euler equation: $e^{ix} = \cos x + i \sin x \qquad i^2 = -1$



Discrete Fourier transform (Matlab / linear algebra)

N samples

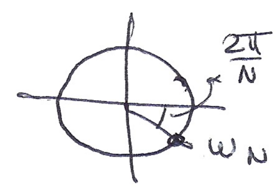




time domain f $\rightleftarrows$ frequency domain $\hat f$

$$\begin{bmatrix} \hat f_0 \\ \hat f_1 \\ \hat f_2 \\ \vdots \\ \hat f_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix}, \quad \omega_N = e^{-\frac{2\pi i}{N}}$$

Fourier coeffs DFT matrix data

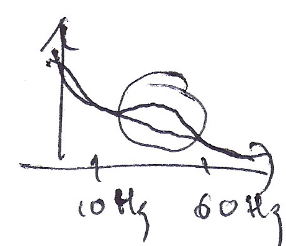


N^2 costly

Fast Fourier transform

$N \log N$ efficient

"fft"



"ifft"

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \cdots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \cdots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \cdots & \end{bmatrix} \begin{bmatrix} \hat f_0 \\ \hat f_1 \\ \hat f_2 \\ \vdots \\ \hat f_{N-1} \end{bmatrix}$$

data inverse DFT matrix Fourier coeff.

LFP : large field potential

delta
theta
gamma

population average of neural activity