Linear algebra

NEU 466M

Spring 2020

Notation

- Matrices: upper-case

  A, B, U, W

- Vector: **bold**, (usually) lower-case

  $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w}$ (handwriting: $\mathbf{x} \rightarrow \underline{x}$)

- Elements of matrix, vector: lower-case

  a_{ij}, b_i, v_j, u_{kl}

- Scalar numbers: lower-case, no indices

  a, b, c, γ, α

Vectors and matrices

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

size *(m x 1)* column vector

$$v_i \in \mathbb{R}$$
$$\mathbf{v} \in \mathbb{R}^m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

size *(n x m)* matrix

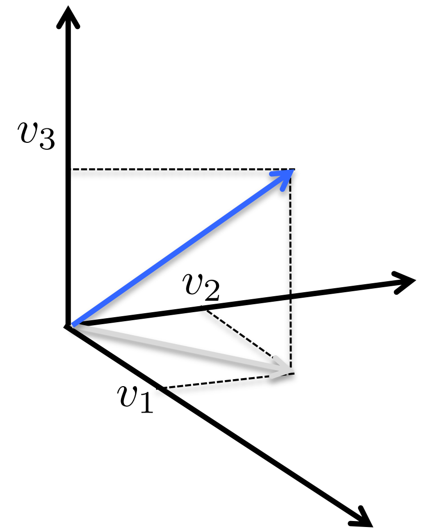
$$A \in \mathbb{R}^{n \times m}$$

What is a vector?

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

size *(m x 1)* column vector

geometric view



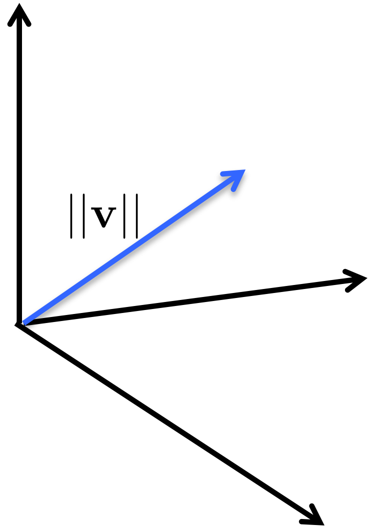
`plotv` in matlab

Vector length

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Length (norm):

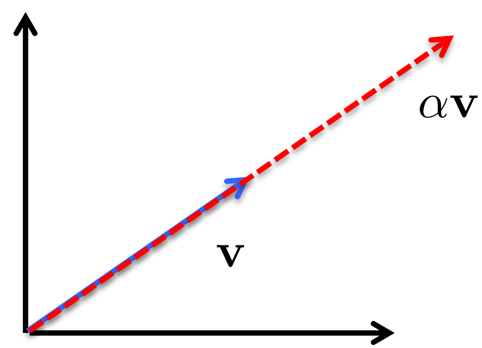
$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots v_m^2}$$



Vector-scalar product

$$\alpha \mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_m \end{bmatrix}$$

geometric view

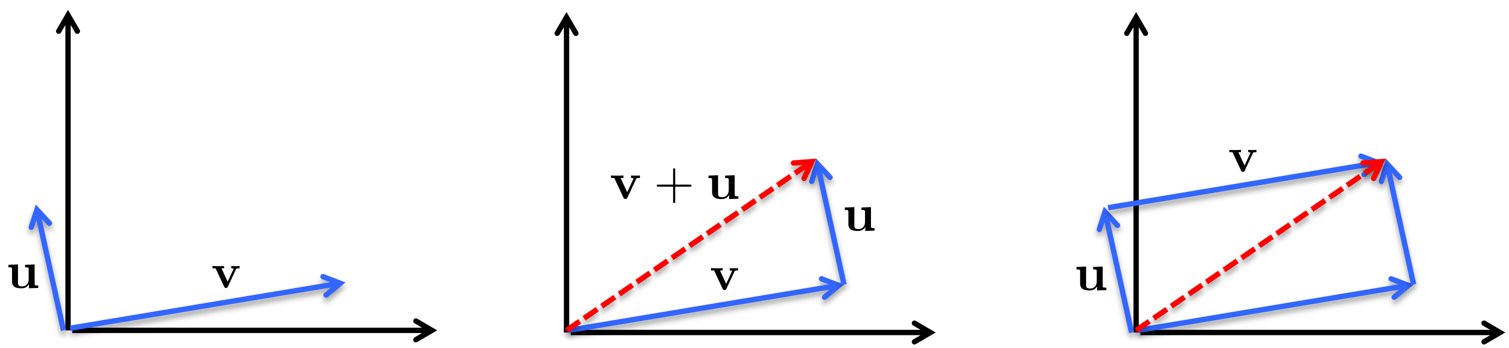


same direction, different length

Sum of vectors

$$\mathbf{v}, \mathbf{u} \in \mathbb{R}^m \qquad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_m + u_m \end{bmatrix}$$

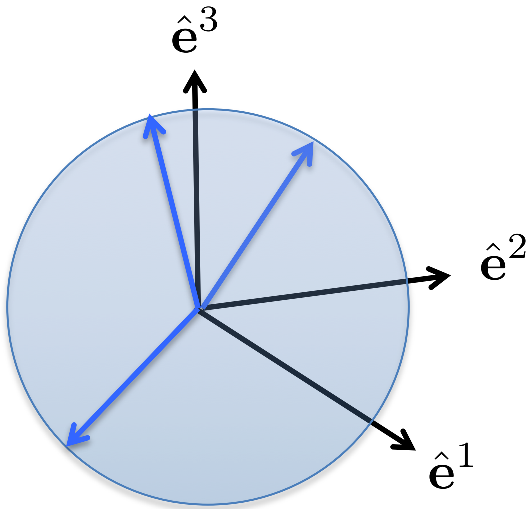
geometric view



Adding vectors: stacking them end-to-end

Unit vector: any vector of length 1

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots v_m^2} = \sqrt{\sum_{i=1}^{m} v_i^2} = 1$$



Every point on *(m-1)*-dimensional sphere of unit radius in *m*-dim space is a unit vector

Vector, matrix transpose

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

size *(m x 1)* column vector

$$\mathbf{v}^T = [v_1 v_2 \cdots v_m]$$

size *(1 x m)* row vector

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

size *(n x m)* matrix

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix}$$

size *(m x n)* matrix

Vector norm as an inner product

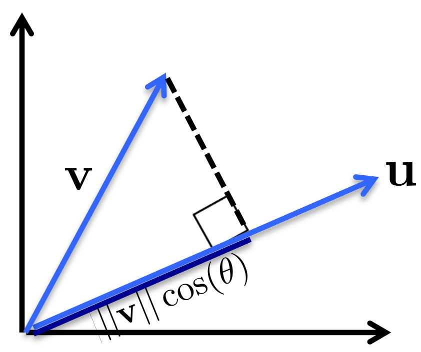
$$\mathbf{v}^T\mathbf{v} = [v_1 v_2 \cdots v_m] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \sum_{i=1}^{m} v_i^2 = ||\mathbf{v}||^2$$

Inner product (dot product)

$$\mathbf{v}, \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{u}^T\mathbf{v} = [u_1 \;\; u_2 \cdots u_m] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \sum_i u_i v_i$$

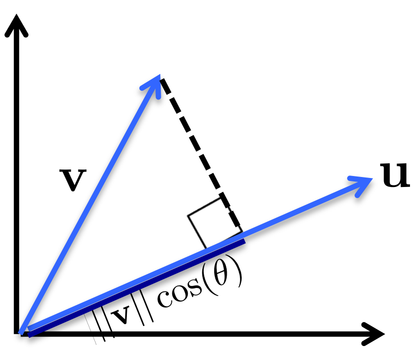
Geometric view: projection of **v** on **u**, times norm of **u:** $\mathbf{u}^T\mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$



Inner product for projection

$$\mathbf{v}, \mathbf{u} \in \mathbb{R}^m$$

If **u** is a unit vector, then inner product the projection of **v** on **u:** $\mathbf{u}^T\mathbf{v} = ||\mathbf{v}|| \cos(\theta)$

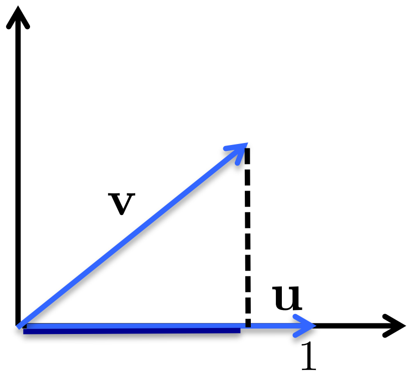


Example: inner product

$$\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$$

Example: $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ unit vector along *x*-axis, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\mathbf{u}^T \mathbf{v} = v_1$$

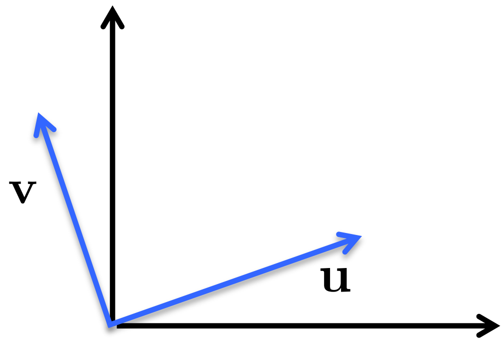


Example: inner product

$$\mathbf{v}, \mathbf{u} \in \mathbb{R}^m$$

Example: $\mathbf{u} \perp \mathbf{v}$

$$\mathbf{u}^T\mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta) = 0$$



Matrix-vector product as inner-product

$$C = AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

$$\mathbf{Av} = \begin{bmatrix} {}^{(2)}a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} {}^{(1)}a_{11}v_1 + a_{12}v_2 + \cdots + a_{1m}v_m \\ {}^{(2)}a_{21}v_1 + a_{22}v_2 + \cdots + a_{2m}v_m \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nm}v_m \end{bmatrix}$$

(n x m) *(m x 1)* *(n x 1)*

$$(\mathbf{Av})_i = \sum_{j=1}^{m} a_{ij} v_j$$

sum notation
i: any index in *{1,…,n}*

$$= \mathbf{a}_i^T \mathbf{v}$$

inner-product between
***v** and ith* row of *A*.

System of equations

n equations in *m* unknowns $(v_1,\ldots v_m)$:

$$a_{11}v_1 + \cdots + a_{1m}v_m = b_1$$
$$a_{21}v_1 + \cdots + a_{2m}v_m = b_2$$
$$\cdots\cdots\cdots$$
$$a_{n1}v_1 + \cdots + a_{nm}v_m = b_n$$

System of equations

n equations in *m* unknowns $(v_1, \ldots v_m)$:

$$a_{11}v_1 + \cdots + a_{1m}v_m = b_1$$
$$a_{21}v_1 + \cdots + a_{2m}v_m = b_2$$
$$\cdots\cdots\cdots$$
$$a_{n1}v_1 + \cdots + a_{nm}v_m = b_n$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(n x m) *(m x 1)* *(n x 1)*

$$A\mathbf{v} = \mathbf{b}$$

System of equations: when does unique solution exist?

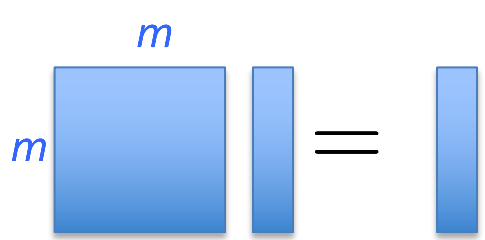
n equations in *m* unknowns: *generically*, a unique solution exists when same number of constraints *(n)* as unknowns *(m)*: Thus, *n=m or A* is square.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(m x m) *(m x 1)* *(n x 1)*

$$A\mathbf{v} = \mathbf{b}$$

(m x m) *(m x 1)* *(m x 1)*



this is an
algebraic view.
time for some geometric
insight.

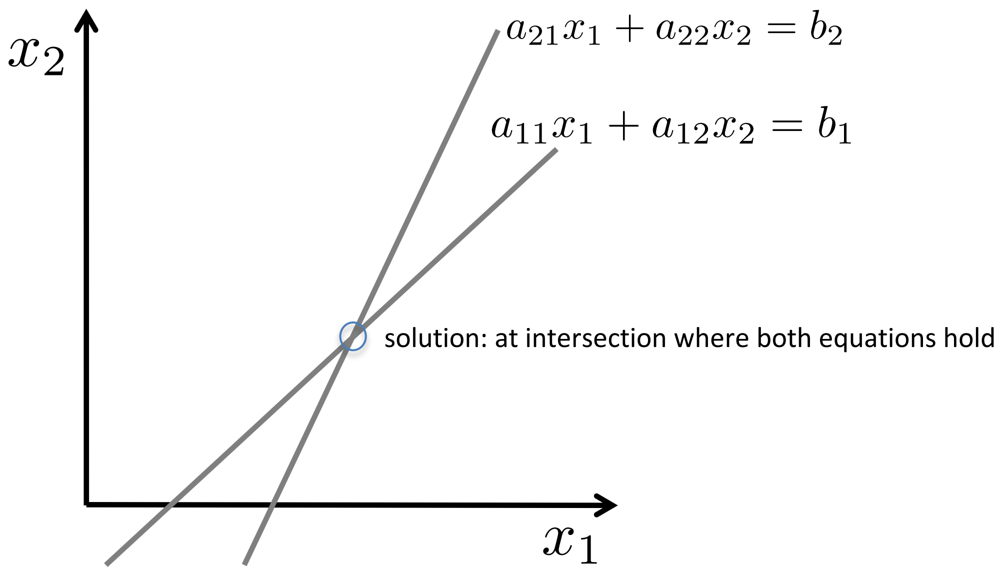
Geometric view: when does a unique solution exist?

Start with 2-dimensional problem: 2 unknowns, 2 equations

equation of a line

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

unknowns x_1, x_2



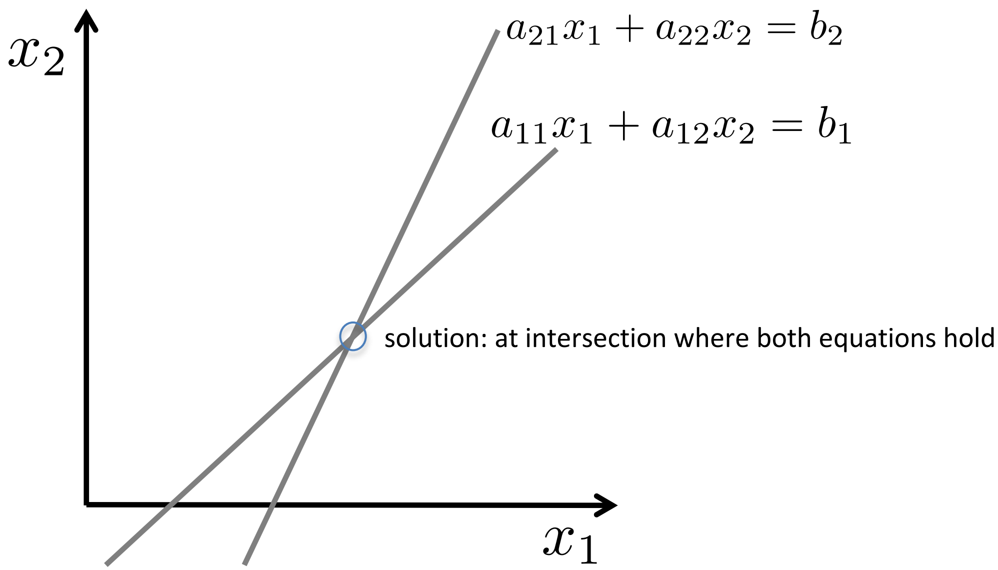
Geometric view: when does a unique solution exist?

Start with 2-dimensional problem: 2 unknowns, 2 equations

equation of a line

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

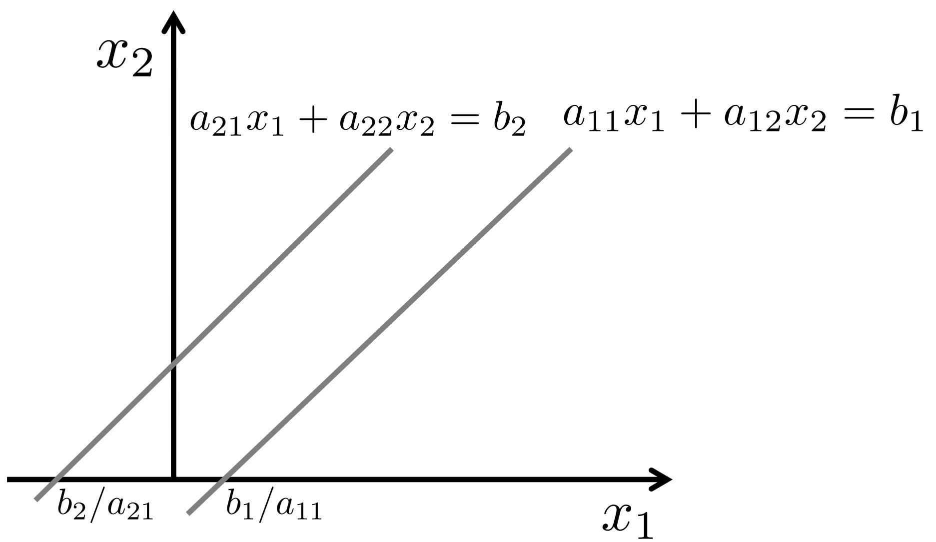
unknowns x_1, x_2



Generically two infinite lines in 2D space intersect at a (single) location thus (unique) solution exists.

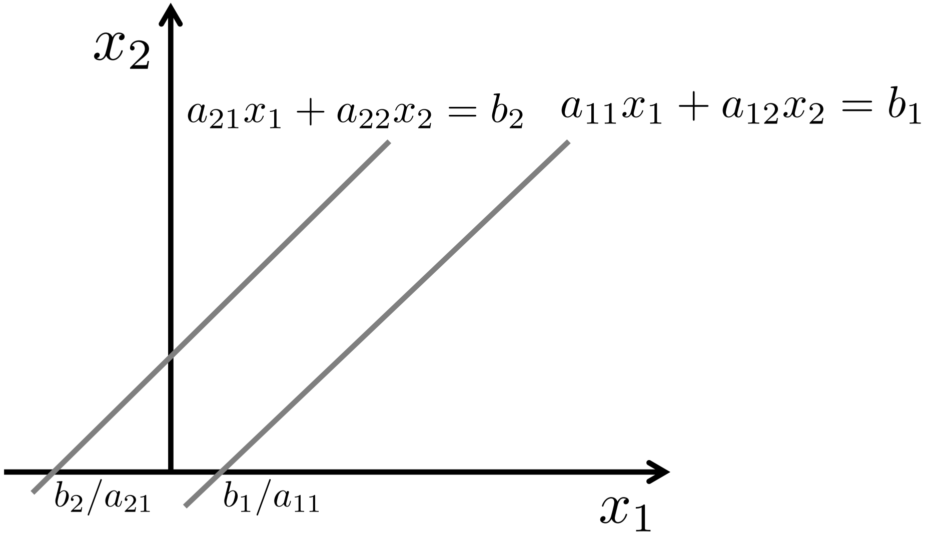
Geometric view: when does a unique solution *not* exist?

1. Offset parallel lines: no solution exists



Algebra: when does a unique solution *not* exist?

1. Offset parallel lines: no solution exists



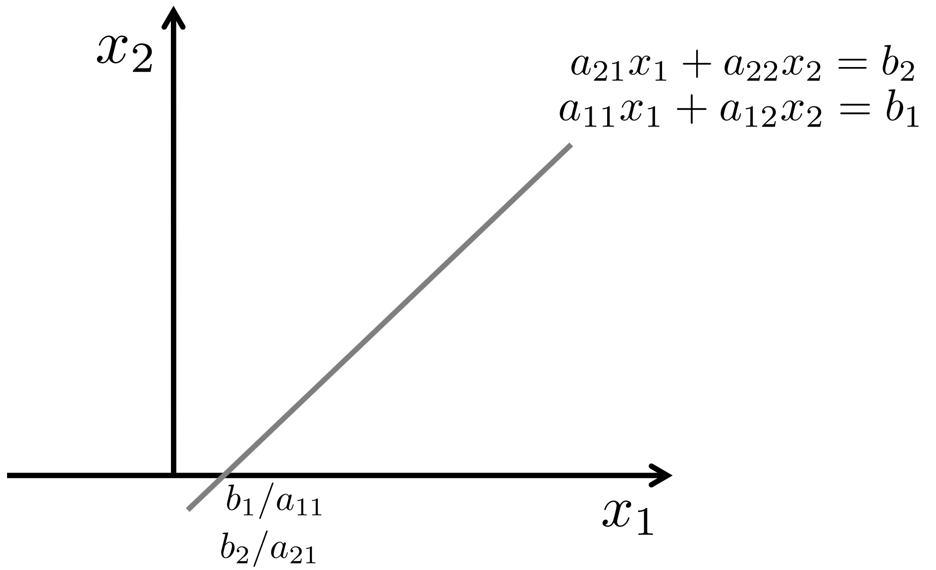
$$a_{21}/a_{22} = a_{11}/a_{12}$$ equal slopes

$$a_{11}a_{22} = a_{12}a_{21}$$

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

Algebra: when does a unique solution *not* exist?

2. Aligned parallel lines: infinitely many solutions



$a_{11}a_{22} - a_{12}a_{21} = 0$ equal slopes

$b_1/a_{11} = b_2/a_{21}$ equal intercepts

Algebraic view: existence of unique solution in terms of coefficient matrix *A*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

determinant: $$\det(A) \equiv a_{11}a_{22} - a_{12}a_{21}$$

2-dim system of equations with square coefficient matrix *A* has a unique solution when:

$$\det(A) \neq 0$$

Same condition for *m*-dim system of equations with square coefficient matrix.

Linear system: possibilities

- 1 unique solution
- No solutions
- Infinitely many solutions