Wiener-Hopf kernel estimation

NEU 466M

Spring 2020

Problem setup

$$\{\cdots x_{t-1}, x_t, x_{t+1}\cdots\}$$
 time-varying signal (stimulus) sampled at discrete intervals $\{\cdots y_{t-1}, y_t, y_{t+1}\cdots\}$ time-varying signal (response)

Assume y derived from x, through convolution with unknown kernel h and small noise term ε :

$$y(n) = \sum_{m=M_1}^{M_2} x(n-m)h(m) + \epsilon(n)$$

$$\{h_{M_1},\cdots,h_{M_2}\}$$
 unknown kernel. If $\mathsf{M_1}$ =0: causal.

Question

$$y(n) = \sum_{m=M_1}^{M_2} x(n-m)h(m) + \epsilon(n)$$

What is the best possible estimate of h, given x, y?

Minimization problem

$$E = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right)^2$$

$$\hat{h}_{M_1} \cdots \hat{h}_{M_2} = \arg \min_{h_{M_1} \cdots h_{M_2}} E$$

Find h that minimizes the squared error between measured y and values predicted from x by the model.

Minimization problem

$$E = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right)^2$$

$$\hat{h}_{M_1} \cdots \hat{h}_{M_2} = \arg \min_{h_{M_1} \cdots h_{M_2}} E$$

Compare: (very similar) linear regression framework.

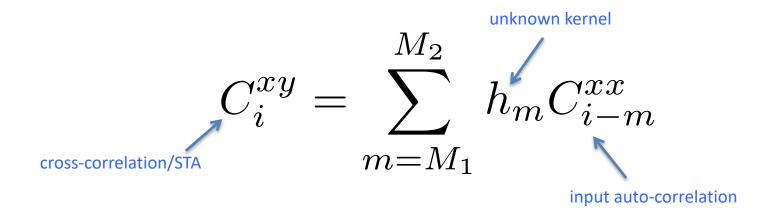
Solve: minimization problem

$$0 = \frac{\partial E}{\partial h_i} = -\sum_{n} \left(y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right) x_{n-i}$$

$$= -\sum_{n} y_n x_{n-i} + \sum_{m} (h_m \sum_{n} x_{n-m} x_{n-i})$$

$$= C_i^{xy} - \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

Wiener-Hopf equations



- This is the least-squares optimal solution for the unknown kernel h.
- It depends on the cross-correlation of the input and the response (STA).
- But it also depends on the auto-correlation of the input, unlike the STA.

Linear regression as special case of Wiener-Hopf

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

No time-lags in auto- and cross-correlation since x,y independent samples not time series (so i=0). Only one term h (the slope between x,y), no convolution, so $M_1=M_2=0$

$$C^{xy} = C^{xx}h_0$$

$$h_0 = \frac{C^{xy}}{C^{xx}}$$

Optimal least-squares estimate of slope in linear regression (look back at notes)

Wiener-Hopf equations: solution?

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m^x C_{i-m}^{xx}$$
 input auto-correlation

 M_2 - M_1 +1 unknowns h_m .

 M_2 - M_1 +1 equations: i^{th} equation obtained by differentiating w.r.t. h_i .

Thus, generically, a solution exists.

Easy way to solve?