Wiener-Hopf kernel estimation

NEU 466M

Spring 2020

Problem setup

$$\{\cdots x_{t-1}, x_t, x_{t+1} \cdots\}$$ time-varying signal (stimulus) sampled at discrete intervals

$$\{\cdots y_{t-1}, y_t, y_{t+1} \cdots\}$$ time-varying signal (response)

Assume *y* derived from *x,* through convolution with unknown kernel *h* and small noise term *ε:*

$$y(n) = \sum_{m=M_1}^{M_2} x(n-m)h(m) + \epsilon(n)$$

$$\{h_{M_1}, \cdots, h_{M_2}\}$$ unknown kernel. If M_1=0: causal.

Question

$$y(n) = \sum_{m=M_1}^{M_2} x(n-m)h(m) + \epsilon(n)$$

What is the best possible estimate of *h*, given *x, y*?

Minimization problem

$$E = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right)^2$$

$$\hat{h}_{M_1} \cdots \hat{h}_{M_2} = \arg \min_{h_{M_1} \cdots h_{M_2}} E$$

Find *h* that minimizes the squared error between measured *y* and values predicted from *x* by the model.

Minimization problem

$$E = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right)^2$$

$$\hat{h}_{M_1} \cdots \hat{h}_{M_2} = \arg \min_{h_{M_1} \cdots h_{M_2}} E$$

Compare: (very similar) linear regression framework.

Solve: minimization problem

$$0 = \frac{\partial E}{\partial h_i} = -\sum_n \left(y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right) x_{n-i}$$

$$= -\sum_n y_n x_{n-i} + \sum_m (h_m \sum_n x_{n-m} x_{n-i})$$

$$= C_i^{xy} - \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

Wiener-Hopf equations

unknown kernel

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

cross-correlation/STA

input auto-correlation

- This is the least-squares optimal solution for the unknown kernel *h.*
- It depends on the cross-correlation of the input and the response (STA).
- But it also depends on the auto-correlation of the input, unlike the STA.

Linear regression as special case of Wiener-Hopf

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

No time-lags in auto- and cross-correlation since *x,y* independent samples not time series (so *i=0*). Only one term *h* (the slope between *x, y*), no convolution, so $M_1 = M_2 = 0$

$$C^{xy} = C^{xx} h_0$$

$$h_0 = \frac{C^{xy}}{C^{xx}}$$

Optimal least-squares estimate of slope in linear regression (look back at notes)

Wiener-Hopf equations: solution?

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

unknown kernel: $(M_2\text{-}M_1\text{+}1)$ x 1

$(M_2\text{-}M_1\text{+}1)$ x 1 cross-correlation/STA

input auto-correlation

M_2-M_1+1 unknowns h_m.
M_2-M_1+1 equations: i^{th} equation obtained by differentiating w.r.t. h_i.
Thus, generically, a solution exists.
Easy way to solve?