Wiener-Hopf kernel estimation

NEU 466M

Spring 2020

Problem setup

$$\{\cdots x_{t-1}, x_t, x_{t+1} \cdots\}$$ time-varying signal (stimulus) sampled at discrete intervals

$$\{\cdots y_{t-1}, y_t, y_{t+1} \cdots\}$$ time-varying signal (response)

Assume *y* derived from *x*, through convolution with unknown kernel *h* and small noise term *ε:*

$$y(n) = \sum_{m=M_1}^{M_2} x(n-m)h(m) + \epsilon(n)$$

$$\{h_{M_1}, \cdots, h_{M_2}\}$$ unknown kernel.
If M_1=0: causal.

Wiener-Hopf equations

unknown kernel

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

cross-correlation/STA

input auto-correlation

- This is the least-squares optimal solution for the unknown kernel *h*.
- It depends on the cross-correlation of the input and the response (STA).
- But it also depends on the auto-correlation of the input, unlike the STA.

Linear regression as special case of Wiener-Hopf

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

No time-lags in auto- and cross-correlation since *x,y* independent samples not time series (so *i=0*). Only one term *h* (the slope between *x, y*), no convolution, so $M_1 = M_2 = 0$

$$C^{xy} = C^{xx} h_0$$

$$h_0 = \frac{C^{xy}}{C^{xx}}$$

Optimal least-squares estimate of slope in linear regression (look back at notes)

Wiener-Hopf equations: solution?

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

unknown kernel: $(M_2\text{-}M_1\text{+}1)$ x 1

$(M_2\text{-}M_1\text{+}1)$ x 1 cross-correlation/STA

input auto-correlation

M_2-M_1+1 unknowns h_m.
M_2-M_1+1 equations: i^{th} equation obtained by differentiating w.r.t. h_i.
Thus, generically, a solution exists.
Easy way to solve?

Brief algebra detour before we solve Wiener-Hopf equations

MATRIX-VECTOR ALGEBRA

Matrix-vector algebra

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

size *(n x m)* matrix

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

size *(m x 1)* column vector

Notation

- Matrices: upper-case

  A, B, U, W

- Vector: **bold**, (usually) lower-case

  $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w}$ (handwriting: $\mathbf{x} \rightarrow \underline{x}$)

- Elements of matrix, vector: lower-case

  a_{ij}, b_i, v_j, u_{kl}

- Scalar numbers: lower-case, no indices

  a, b, c, γ, α

Matrix-vector algebra

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1m}v_m \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2m}v_m \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nm}v_m \end{bmatrix}$$

(1) (2)

(n x m) *(m x 1)* *(n x 1)*

$$(\mathbf{A}\mathbf{v})_i = \sum_{j=1}^{m} a_{ij} v_j$$

i any index in *{1,…,n}*

Matrix-matrix product

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & b_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{ml} \end{bmatrix}$$

(n x m) *(m x l)*

$$= \begin{bmatrix} (a_{11}b_{11} + \cdots + a_{1m}b_{m1}) & \cdots & (a_{11}b_{1l} + \cdots + a_{1m}b_{ml}) \\ (a_{21}b_{11} + \cdots + a_{2m}b_{m1}) & \cdots & (a_{21}b_{1l} + \cdots + a_{2m}b_{ml}) \\ \cdots & \cdots & \cdots \\ (a_{n1}b_{11} + \cdots + a_{nm}b_{m1}) & \cdots & (a_{n1}b_{1l} + \cdots + a_{nm}b_{ml}) \end{bmatrix}$$

(n x l)

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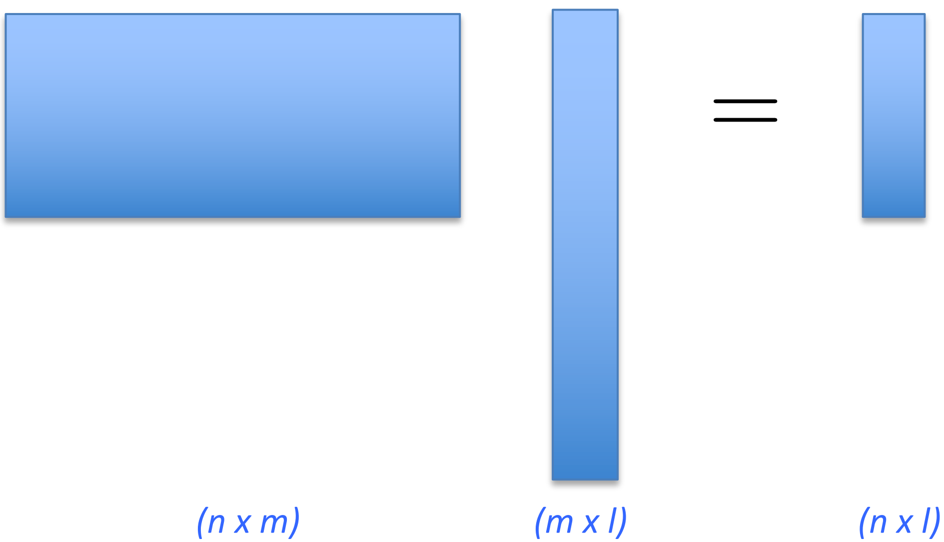
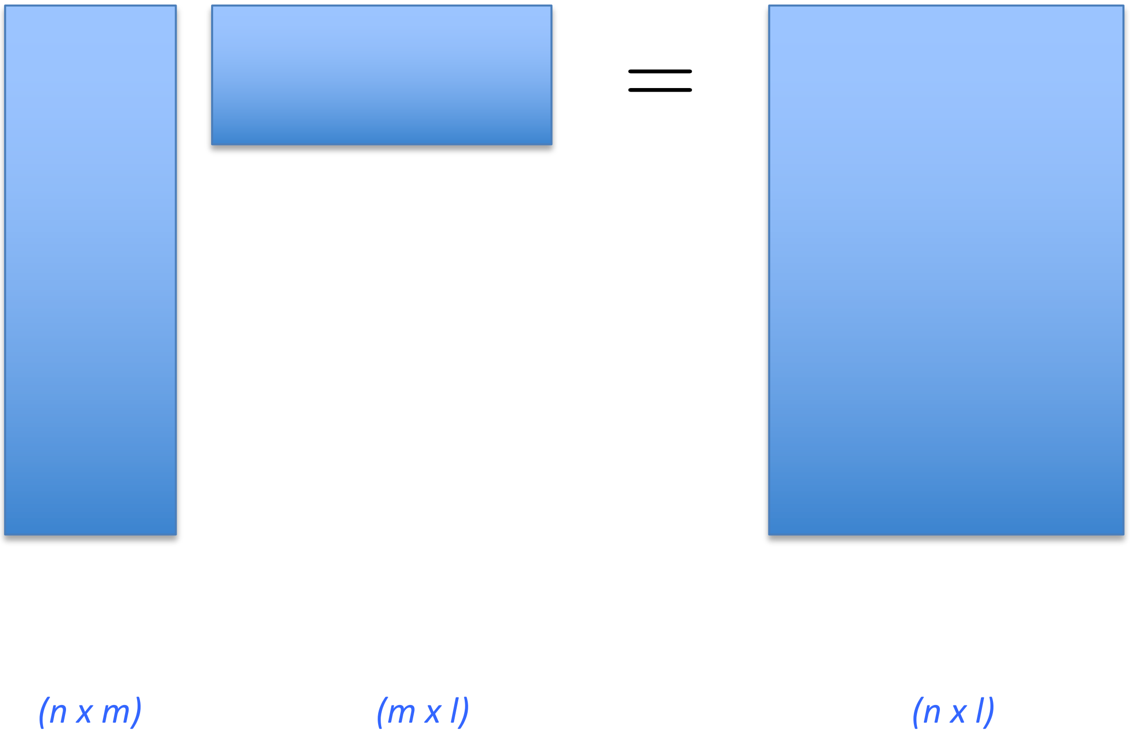
Matrix-matrix product

$$AB =$$



Matrix-matrix product

$$AB =$$



System of equations

n equations in *m* unknowns $(v_1, \ldots v_m)$:

$$a_{11}v_1 + \cdots + a_{1m}v_m = b_1$$
$$a_{21}v_1 + \cdots + a_{2m}v_m = b_2$$
$$\cdots\cdots\cdots\cdot$$
$$a_{n1}v_1 + \cdots + a_{nm}v_m = b_n$$

System of equations

n equations in *m* unknowns $(v_1, \ldots v_m)$:

$$
\begin{aligned}
a_{11}v_1 + \cdots + a_{1m}v_m &= b_1 \\
a_{21}v_1 + \cdots + a_{2m}v_m &= b_2 \\
&\cdots\cdots\cdots \\
a_{n1}v_1 + \cdots + a_{nm}v_m &= b_n
\end{aligned}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}
\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

(n x m) *(m x 1)* *(n x 1)*

System of equations

n equations in *m* unknowns $(v_1, \ldots v_m)$:

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$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(n x m) *(m x 1)* *(n x 1)*

$$A\mathbf{v} = \mathbf{b}$$

System of equations: when does unique solution exist?

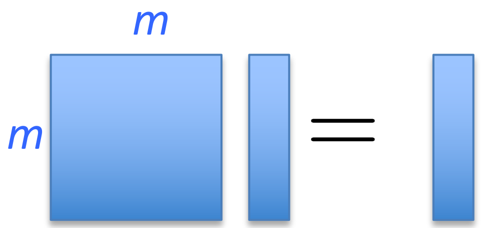
n equations in *m* unknowns: *generically*, a unique solution exists when same number of Constraints *(n)* as unknowns *(m)*: *n=m or A* is square

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(m x m) *(m x 1)* *(n x 1)*

$$A\mathbf{v} = \mathbf{b}$$

(m x m) *(m x 1)* *(m x 1)*



System of equations: when does unique solution exist?

n equations in *m* unknowns*: generically,* a unique solution exists when *n=m,* or *A* is square

$$A\mathbf{v} = \mathbf{b}$$

(n x n) *(n x 1)* *(n x 1)*

When solution exists, it is given by:

$$\mathbf{v} = A^{-1}\mathbf{b}$$

(n x 1) *(n x n)* *(n x 1)*

Where A^{-1} is the inverse of the matrix A, and is defined as:

$$A^{-1}A = AA^{-1} = I$$

Identity matrix

Identity matrix

(n x m) (m x m) *(n x m)*

$$BI = B$$

$$IB = B$$

(n x n) (n x m) *(n x m)*

Square matrix with *1*'s on the diagonal*, 0*'s everywhere else*:*

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Return to

SOLUTION OF WIENER-HOPF EQUATIONS

Wiener-Hopf equations: matrix form

$$C_i^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx}$$

define matrix C^{xx} such that $(C^{xx})_{i,m} \equiv C_{i-m}^{xx}$ — size $(M_2-M_1+1) \times (M_2-M_1+1)$

vector $\mathbf{C}^{xy}$ such that $(\mathbf{C}^{xy})_i \equiv C_i^{xy}$ — size $(M_2-M_1+1) \times 1$

vector $\mathbf{h}$ such that $(\mathbf{h})_i \equiv h_i$ — size $(M_2-M_1+1) \times 1$

$$(\mathbf{C}^{xy})_i = \sum_{m=M_1}^{M_2} (C^{xx})_{i,m} h_m$$

$$= (C^{xx}\mathbf{h})_i$$

$$\mathbf{C}^{xy} = C^{xx}\mathbf{h}$$

Wiener-Hopf equations in matrix-vector form

Solution of the Wiener-Hopf equations

$$\mathbf{C}^{xy} = C^{xx}\mathbf{h}$$

(M_2-M_1+1) x (M_2-M_1+1) inverse auto-correlation matrix

$$\mathbf{h} = (C^{xx})^{-1}\mathbf{C}^{xy}$$

unknown kernel: (M_2-M_1+1) x 1

STA: (M_2-M_1+1) x 1

Matlab: `toeplitz` for autocorrelation matrix, `A\b` for $A^{-1}b$

The autocorrelation matrix

$$C^{xx} = \begin{bmatrix} C_0^{xx} & C_1^{xx} & C_2^{xx} & \cdots & C_K^{xx} \\ C_1^{xx} & C_0^{xx} & C_1^{xx} & & \\ \vdots & \ddots & \ddots & \ddots & \\ C_{K-1}^{xx} & & \ddots & C_0^{xx} & C_1^{xx} \\ C_K^{xx} & C_{K-1}^{xx} & & C_1^{xx} & C_0^{xx} \end{bmatrix}$$

$K=M_2-M_1+1$
square
Toeplitz

The autocorrelation matrix

$$C^{xx} = \begin{bmatrix} C_0^{xx} & C_1^{xx} & C_2^{xx} & \cdots & C_K^{xx} \\ C_1^{xx} & C_0^{xx} & C_1^{xx} & & \\ \vdots & \ddots & \ddots & \ddots & \\ C_{K-1}^{xx} & & \ddots & C_0^{xx} & C_1^{xx} \\ C_K^{xx} & C_{K-1}^{xx} & & C_1^{xx} & C_0^{xx} \end{bmatrix}$$

$K=M_2-M_1+1$
square
Toeplitz

When stimulus *x* is white noise then autocorrelation zero everywhere except at 0-lag.
Thus, $C^{xx}_{ij} = 0$ except along main diagonal → $C^{xx}_{i\,j} = I$ *(identity matrix).*

Wiener-Hopf solution when stimulus is white

$$\mathbf{h} = (C^{xx})^{-1}\mathbf{C}^{xy}$$

$$C^{xx}, (C^{xx})^{-1} = I$$ for white noise stimulus

$$\mathbf{h} = \mathbf{C}^{xy} = STA$$

The Wiener-Hopf estimate of the kernel is the STA when the stimulus is uncorrelated.

Summary

- Wiener-Hopf equations give the (least-squared error) optimal estimate of an unknown kernel between input x and response *y.*
- Linear regression is special case of Wiener-Hopf filtering for stationary (non time-series) data.
- STA is a special case of the Wiener-Hopf kernel if the stimulus is white. Thus, STA is the best (minimum squared-error) kernel estimate for uncorrelated stimulus.
- For positively correlated stimulus, STA kernel estimate is always wider than the true kernel. Wiener-Hopf solution: normalize kernel by inverse of stimulus correlation matrix (this accounts for the stimulus-induced response correlation).