Quantitative Methods in Neuroscience (NEU466M)
Homework 1
Due: Thursday January 30 by 12 pm (upload on canvas).

This homework introduces you to some interesting, special matrices, giving some feel for what matrices do to vectors. *General guidelines:* Read through each complete problem carefully before attempting any parts. Feel free to collaborate in groups of size 2-3, but always note the names of your collaborators on your submitted homework. For graphs: clearly label your axes and use good color and symbol choices. Print out your matlab code (in the form of a script file). For derivations you're asked to do 'by hand' (in other words, analytically, using paper and pencil) feel free to turn in handwritten or typed-out work.

1) **Unit vectors, angle between vectors, and vector projection.**

a. Derive the unit vectors $\mathbf{e_u}, \mathbf{e_v}$ in the direction of $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

b. Derive the inner angle between $\mathbf{u}$ and $\mathbf{v}$. Show your work.

c. Express $\mathbf{u}$ with a different set of basis vectors. In a. above, we wrote $\mathbf{u}$ in the standard basis of $\hat{\mathbf{e}}_\mathbf{1} = (\mathbf{1}, \mathbf{0}), \hat{\mathbf{e}}_\mathbf{2} = (\mathbf{0}, \mathbf{1})$ as $\mathbf{u} = 3\hat{\mathbf{e}}_\mathbf{1} + 1\hat{\mathbf{e}}_\mathbf{2}$. Thus, the *coefficients* or *projections* of $\mathbf{u}$ onto that standard basis are $3, 1$, respectively. The new basis is $\hat{\mathbf{e}}'_\mathbf{1} = \frac{1}{\sqrt{2}}(1, 1), \hat{\mathbf{e}}'_\mathbf{2} = (-1, 1)$. Is this an orthogonal basis? Are the basis vectors normalized (unit length)? First draw by hand the old basis, the new basis, and the vector $\mathbf{u}$. Also by hand on the same plot, show how to project $\mathbf{u}$ onto the old and new bases. Derive the coefficients of $\mathbf{u}, \mathbf{v}$ in the new basis. Show your work.

2) **Some special matrices, plotting vectors in Matlab.**

a. Consider the 2×2 matrix $M(\theta)$, given by

$$M(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

for some choice of θ (you choose). In Matlab, multiply M into the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plot $\mathbf{v}$ and $\mathbf{v}' = M\mathbf{v}$ on the same plot. (Hint: use `plotv`) Next generate $\mathbf{v}'' = M\mathbf{v}'$ (verify that and plot that on top of the previous plot. (Note that $\mathbf{v}', \mathbf{v}''$ do not refer to transposes of $\mathbf{v}$ here, even though primes denote vector and matrix transposition in Matlab; here they are simply the names of different vectors.) Repeat for various different choices of θ. State precisely how $\mathbf{v}, \mathbf{v}'$ and $\mathbf{v}''$ differ and in what ways they remain the same. Next, compute the product $M\mathbf{v}$ by hand, for general θ. What can you conclude about the effect of $M(\theta)$ on vectors? What might you want to call the matrix M? Finally, compute $(M(\theta))^2$ in terms of the variable θ, and rewrite each element as just a cosine or a sine, using the double-angle/half-angle formulae for sines and cosines. What does applying M repeatedly do to a vector?

b. Consider another 2×2 matrix R, given by

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Right-multiply R by hand (analytically) with an arbitrary vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, to generate $\mathbf{v}' = R\mathbf{v}$. Predict what will happen if you repeat the process, and multiply $\mathbf{v}'$ by R? Do not do this in matlab. Instead, use the observation that $\mathbf{v}'' = R\mathbf{v}' = R^2\mathbf{v}$, and by hand (not in Matlab), compute R^2.

Summarize the operation that R performs on vectors. What might you want to call it?

c. Consider the 2×2 matrix A, given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and consider the column vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. By hand, compute $\mathbf{y} = A\mathbf{v}$ and $\mathbf{z}^T = \mathbf{v}^T A$. Is $\mathbf{y} = \mathbf{z}$? If not, what condition on A will guarantee that they are? This is the definition of a symmetric matrix.

3) **Orthogonal matrices**

a. Length-preserving matrices: Derive the conditions that an arbitrary-dimension square matrix A must obey, to preserve the norms of all vectors it acts on. (Hint: recall that the squared norm of a real-valued column vector $\mathbf{x}$ is given by $\mathbf{x}^T\mathbf{x}$.) This is the definition of an orthogonal matrix.

b. Are the matrices $M(\theta)$ and R above orthogonal matrices?

c. Determinant: As we have already seen, an important scalar quantity associated with a square matrix is its determinant. Compute the determinant of the matrices $M(\theta)$ and R above (without substituting in a specific value of θ). Orthogonal matrices always have determinant equal to ± 1.

d. Here's another orthogonal matrix:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Verify by hand that it's orthogonal, show your work. Figure out what it does, and explain.