Vectorial spaces, matrix representation, special matrices

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Algebraic operations on matrices can be interpreted geometrically if one considers coefficients of matrices as coordinates of vectors. However, our geometrical intuition suggests that vectors "exist" in their own right irrespective of their coordinates. After all, the same vector may have different coordinates in different coordinate systems. This means that the link between vectorial quantities and matrices is bit more subtle than at first look. This note is to clarify the relationship between linear algebra, which deals with operation on vectors, and matrix algebra, which deals with operation on vectors coordinates in a basis system.

1 Vector spaces

Vectors are meant to be multiplied by numbers belonging to a field—typically the field of real numbers $\mathbb{R}$—and added together in order to form linear combinations. For instance, our geometrical intuition confirms that given two real numbers α and β, two plane vectors $\vec{u}$ and $\vec{v}$ can be linearly combined to form a new plane vector $\alpha\vec{u} + \beta\vec{v}$. Vectorial space are defined as sets of vectors which are stable by linear combinations: if two vectors $\vec{v}$ and $\vec{w}$ belong to a vectorial space V, we are guaranteed that any linear combinations $\alpha\vec{u} + \beta\vec{v}$, $\alpha, \beta \in \mathbb{R}$, also belongs to the space V. Observe that this definition of a vector space does not require to mention a coordinate system or to list vectorial components. All is required is that linear combinations of vectors of V make sense, i.e. are themselves vectors of V.

Suppose now that we are given a single non-zero vector $\vec{v} \neq 0\vec{v} = \vec{0}$ in V. The only linear combinations we can form are by multiplication by a real number α. Geometrically, the set of vectors obtained by such an operation lies onto a straight line directed by $\vec{v}$ and going through the origin $\vec{0}$. We call that set of vector the span of $\vec{v}$

$$\mathrm{span}(\vec{v}) \stackrel{\mathrm{def}}{=} \{\alpha\vec{v} | \alpha \in \mathbb{R}\} . \tag{1}$$

Observe that adding any two vectors from $\mathrm{span}(\vec{v})$ gives us another vector from $\mathrm{span}(\vec{v})$. This is the simplest example of a non-zero vectorial space, which is also a subset of V.

What happens if we consider an additional non-zero vectors $\vec{u}$? Addressing that question leads us consider the span of two vectors defined as

$$\mathrm{span}(\vec{u}, \vec{v}) \stackrel{\mathrm{def}}{=} \{\alpha\vec{u} + \beta\vec{v} | \alpha, \beta \in \mathbb{R}\}, \tag{2}$$

which is a vectorial space in its own right and a subset of V. Two cases need to be distinguished: Either $\vec{u}$ lies along the direction of $\vec{v}$, i.e. $\vec{u} \in \mathrm{span}(\vec{v})$, and all linear combinations of $\vec{u}$ and $\vec{v}$ remains on that straight line, i.e. $\mathrm{span}(\vec{v}) = \mathrm{span}(\vec{u}) = \mathrm{span}(\vec{u}, \vec{v})$. Either $\vec{u}$ does not lies along the direction of $\vec{v}$, i.e. $\vec{u} \notin \mathrm{span}(\vec{v})$, and linear combinations of $\vec{u}$ and $\vec{v}$ defines vectors that span the entire plane defined by $\vec{u}$ and $\vec{v}$, $\mathrm{span}(\vec{u}, \vec{v})$ is strictly larger than $\mathrm{span}(\vec{u})$ and $\mathrm{span}(\vec{v})$. Thus, considering vector $\vec{u}$ in addition to $\vec{v}$ yields a strictly larger span $\mathrm{span}(\vec{u}, \vec{v})$ only if $\vec{u} \notin \vec{v}$. The above results easily generalized to an arbitrary number of vectors. For instance, if we have a family a n vectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, considering a new vector v_{n+1} yields a span $\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_{n+1})$ that is strictly larger than $\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_n)$ only if $v_{n+1} \notin \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_n)$.

Our geometrical intuition suggests that adding a new vector causes the span of the resulting family of vectors to increase if it adds one "dimension" to the resulting vectorial space V. To make the notion of dimension precise, we are going to introduce the notion of free family of vectors. A free family in V is a collection of vectors $\vec{v}_1, \ldots, \vec{v}_n$ such that no vector $\vec{v}_i$ belongs to the span of the other vectors, $\mathrm{span}(\vec{v}_{j \neq i})$. In other words, vectors of a free family are linearly independent because no vector can be expressed as a linear combinations of the others. In particular, we deduce that the larger the free-family, the larger the vectorial space it spans. Can we find ever larger such vectorial space? It turns out that the answer depends on the vectorial space V we are considering. Suppose V is a plane mapped via 2 plane coordinates. It is intuitively clear that no free family with more than three vectors exists. To be linearly independent, a third vector would need to stick out the plane in the ambient 3-dimensional space. Alternatively, if V is our ambient space mapped via 3 space coordinates, it can be shown that no free family with more than four vectors exists. This leads us to define the dimension of a vector space as follows: the dimension of a vector space V, denoted $\dim V$, is the maximal number of vectors in a free family contained in V [1].

To define the dimension of a vectorial space, we have resorted to the notion of free family, which are special families of vectors that are linearly independent

[1] The dimension of a vectorial space may be infinite but we will only concerned ourselves with finite-dimensional vectorial space.

in V. A key property of a free family $\vec{v}_1, \ldots, \vec{v}_n$ is that any vector $\vec{u}$ lying in its span admit a unique representation as a linear combination: there is a unique set of numbers $\alpha_1, \ldots, \alpha_n$ such that $\vec{u} = \alpha_1 \vec{v}_1 + \ldots + \alpha_n \vec{v}_n$. In a finite-dimensional space V, free families of maximal size play a very special role and are given a special name, they are called bases of V. What is special about bases? Their vectors span the entire space V so that every vector in V can be represented by a unique linear combination of basis vectors. In other words, they provide us with a coordinate systems to write down vector components. This is where linear algebra and matrix algebra connects. Given a vectorial space with $\dim V = n$ and a basis $\vec{e}$, one can represent each vector $\vec{v}$ by its set of components in the basis, $\boldsymbol{v} = (v_1, \ldots, v_n)$. These components form the matrix representation of $\vec{v}$ in the basis $\vec{e}$, which we denote as $\boldsymbol{v} = \mathrm{Mat}_{\vec{e}}(\vec{v})$. There is a caveat though: vector components depends on the choice of basis but there is an infinite number of possible choice for bases. In particular, the same vector admits distinct component representations in distinct basis and the geometrical interpretation of matrix algebra always requires to specify a basis of the vectorial space a priori. For the purpose of calculation, not all basis are equal though and by default matrix operation are thought to be carried out in the canonical basis, that is the basis $\vec{e}_1, \ldots, \vec{e}_n$ for which the coordinates of the basis vectors are the matrices $\boldsymbol{e}_1 = (1, 0, \ldots, 0), \boldsymbol{e}_2 = (0, 1, \ldots, 0), \ldots, \boldsymbol{e}_n = (0, \ldots, 1)$.

2 Matrix representation

There is an intimate relationship between matrices and vector spaces. Actually, matrices should be viewed as component representations of vectors, and more generally, as component representations of linear application between vector spaces. This idea is the major source of confusion in linear/matrix algebra and we will discuss that idea in the case of 2 dimensional vectorial spaces for ease of exposition. However, all the arguments generalize to vectorial spaces of arbitrary finite dimension.

Given a basis $\vec{e} = (\vec{e}_1, \vec{e}_2)$, the matrix representation of a vector $\vec{v}$ is the 1-column 1-column of its coordinates in the basis $\vec{e}$. For instance, if $\vec{v} = 2\vec{e}_1 + \vec{e}_2$, we can identify the vector $\vec{v}$ with the matrix:

$$\mathrm{Mat}_{\vec{e}}(\vec{v}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \overset{conv}{=} \boldsymbol{v}\,. \tag{3}$$

For convenience, we denote the matrix representation $\mathrm{Mat}_{\vec{e}}(\vec{v})$ of $\vec{v}$ simply by $\boldsymbol{v}$, where the dependence on the basis is no longer appearing. In a different basis of vectors $\vec{e}' = (\vec{e}'_1, \vec{e}'_2)$, the same vector $\vec{v}$ admits different coordinates, and thus a different matrix representation $\mathrm{Mat}_{\vec{e}\,'}(\vec{v})$, which we denote by $\vec{v}\,'$. A new basis of vectors $\vec{e}\,' = (\vec{e}\,'_1, \vec{e}\,'_2)$ is often defined in relation to an old basis $(\vec{e}_1, \vec{e}_2)$, by specifying the coordinate of the new basis vectors in the old basis. For instance, we may have $\vec{e}\,'_1 = 3\vec{e}_1 + \vec{e}_2$ and $\vec{e}\,'_2 = 2\vec{e}_1 + 3\vec{e}_2$. The coordinates of the new basis in the old basis actually defines a matrix :

$$P_{\boldsymbol{e}}^{\boldsymbol{e}'} = \mathrm{Mat}_{\vec{e}}(\vec{e}\,') = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} . \tag{4}$$

The matrix $P_{\vec{e}\,'}^{\vec{e}}$ is the change of basis matrix from $\vec{e}\,'$ to $\vec{e}$ because it can be interpreted as a change of coordinates. Indeed, in the basis $\vec{e}\,'$, we have

$$\boldsymbol{e}'_1 = \mathrm{Mat}_{\vec{e}\,'}(\vec{e}\,'_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{e}'_2 = \mathrm{Mat}_{\vec{e}\,'}(\vec{e}\,'_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{5}$$

and we can check that right-multiplying the basis vectors $\vec{e}\,'$ by $P_{\boldsymbol{e}}^{\boldsymbol{e}'}$ yields the coordinates of the new basis in the old basis $\vec{e}$:

$$P_{\boldsymbol{e}}^{\boldsymbol{e}'}\boldsymbol{e}'_1 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \mathrm{Mat}_{\vec{e}}(\vec{e}\,'_1) \,, \tag{6}$$

$$P_{\boldsymbol{e}}^{\boldsymbol{e}'}\boldsymbol{e}'_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathrm{Mat}_{\vec{e}}(\vec{e}\,'_2) \,. \tag{7}$$

Thus, for a generic vector $\vec{v}\,' = v'_1\vec{e}\,'_1 + v'_2\vec{e}\,'_2$, the old coordinates $\boldsymbol{v}$ can be obtained from the new ones $\boldsymbol{v}'$ by

$$\begin{aligned}
P_{\boldsymbol{e}}^{\boldsymbol{e}'}\boldsymbol{v}' &= P_{\boldsymbol{e}}^{\boldsymbol{e}'}(v'_1\boldsymbol{e}'_1 + v'_2\boldsymbol{e}'_2) && (8)\\
&= v'_1 P_{\boldsymbol{e}}^{\boldsymbol{e}'}\boldsymbol{e}'_1 + v'_2 P_{\boldsymbol{e}}^{\boldsymbol{e}'}\boldsymbol{e}'_2 && (9)\\
&= v'_1\mathrm{Mat}_{\vec{e}}(\vec{e}\,'_1) + v'_2\mathrm{Mat}_{\vec{e}}(\vec{e}\,'_2) && (10)\\
&= \mathrm{Mat}_{\vec{e}}(v'_1\vec{e}\,'_1 + v'_2\vec{e}\,'_2) && (11)\\
&= \mathrm{Mat}_{\vec{e}}(\vec{v}) && (12)\\
&= \boldsymbol{v} && (13)
\end{aligned}$$

Unfortunately, we are not really interested in the above change of coordinates since we are looking for the new coordinate $\boldsymbol{v}'$ in term of the old ones $\boldsymbol{v}$. Luckily, acheiving this change of coordinates only requires to introduce the inverse of $P_{\boldsymbol{e}}^{\boldsymbol{e}'}$[2], which is the change of basis matrix from $\vec{e}$ to $\vec{e}\,'$, and which satisfies:

$$\boldsymbol{v}' = \left(P_{\boldsymbol{e}}^{\boldsymbol{e}'}\right)^{-1}\boldsymbol{v} = P_{\boldsymbol{e}'}^{\boldsymbol{e}}\boldsymbol{v} \,. \tag{14}$$

[2]Change of basis matrix are always invertible

Thus, we have a simple recipe to find the new coordinates in term of the old coordinates.

We just saw that matrix representations of vectors are 1-column or 1-row matrices listing vectorial coordinates, which depend on the choice of basis. In the following, we focus on square matrices and show that they can be viewed as representation of linear applications f from a vectorial space V into itself. Although outside the scope of this note, this interpretation of matrices as representation of linear functions extends to non-square matrices at the cost of considering linear functions between distinct vectorial spaces, $f : V \to W$, with possibly different dimensions, i.e. such that $\dim V \neq \dim W$.

What are linear applications and how do they relate to matrices? An linear application $f : V \to V$ is a function that takes a vector $\vec{v}$ in V to a vector $f(\vec{v})$ in V while satisfying a linear property. A function f is linear if it maps linear if it "preserves" linear combinations: $f(\alpha\vec{v} + \beta\vec{w}) = \alpha f(\vec{v}) + \beta f(\vec{w})$. An important consequence of the linearity property is that $f(V)$, the output space of f is a vectorial space included in V. Moreover, if we know the image vectors $f(\vec{v}_i)$ of a basis of $\vec{v}_i$, we can express all vectors of $f(V)$ as linear combinations of the $f(\vec{v}_i)$. In other words, we can characterize the action of f in V by focusing on how the application f transform a basis of V. This fact is the essence of the link between linear application and matrices. Actually, we have already seen this link without mentioning it explicitly when discussing change of basis. Indeed, when discussing the right multiplication of the coordinates $\boldsymbol{v}'$ by the matrix $P_{\boldsymbol{e}}^{\boldsymbol{e}'}$—which is a linear operation—, we have interpreted the result $\boldsymbol{v}$ as the coordinates of the same vector but in a different base. In term of linear application, this means that $P_{\boldsymbol{e}}^{\boldsymbol{e}'}$ represents the application that takes a vector $\vec{v}$ to the same vector $\vec{v}$, which the identity application $id : V \to v$. However, the matrix $P_{\boldsymbol{e}}^{\boldsymbol{e}'}$ is clearly different than the identity matrix. The reason for this is that the change of basis matrix is the representation of the identity application $id : V \to V$ when considering the input space equipped with the base $\vec{e}\,'$ and the output space with the base $\vec{e}$.

The above discussion about changes of basis may seem a bit confusing at first sight. However, the message of that discussion is simple. Just as vectors exist in their own rights, linear applications between vectorial spaces also exist in their own rights and admit matrix representations. These matrix representations also depend on the bases choice to represent vectors both in the input and in the output spaces of the linear applications[3]. To make things more concrete, we conclude this discussion by considering a familiar linear transformation: rotations in the 2-dimensional space. In the canonical coordinate of the planes, coordinates of vectors

[3]There are two choices of bases. However, it is often the case that we chose the same basis in the input and output space when possible.

$\vec{v}$ can be parametrized in terms of a radius r and an angle θ with respect to the first basis vector:

$$\boldsymbol{v} = (r\cos(\theta), r\sin(\theta)) \,. \tag{15}$$

A rotation by an angle ψ, denoted Rot_ψ, sends a vector $\vec{v}$ onto an image vector $\vec{v}_\psi = \mathrm{Rot}_\psi(\vec{v})$ with new coordinates $\boldsymbol{v}_\psi = (r\cos(\theta+\psi), r\sin(\theta+\psi))$. Although we express the action of Rot_ψ via coordinates, it is intuitively clear that the relation $\vec{v}_\psi = \mathrm{Rot}_\psi(\vec{v})$ is independent of the choice of coordinates. Moreover, one can show that Rot_ψ is a linear application in the plane.

What are the matrix representations of Rot_ψ in a given base? Let us start in the canonical base $\vec{e}$. Using trigonometric inequalities, we can show that $\boldsymbol{v}_\psi$, the coordinates of the output vector $\vec{v}_\psi$ in the canonical base $\vec{e}$, satisfy

$$\begin{aligned}
\boldsymbol{v}_\psi &= \begin{bmatrix} r\cos(\theta+\psi) \\ r\sin(\theta+\psi) \end{bmatrix} && (16)\\
&= \begin{bmatrix} r\cos(\theta)\cos(\psi) - r\sin(\theta)\sin(\psi) \\ r\sin(\theta)\cos(\psi) + r\cos(\theta)\sin(\psi) \end{bmatrix} && (17)\\
&= \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{R_\psi} \underbrace{\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}}_{\boldsymbol{v}}, && (18)
\end{aligned}$$

where $\boldsymbol{v}$ are the coordinates of the input vector $\vec{v}$ in the canonical base $\vec{e}$. That's it. We just found the matrix representation of $\mathrm{Rot}(\psi)$ in the canonical basis $\vec{e}$:

$$R_\psi = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \mathrm{Mat}_{\vec{e}}(\mathrm{Rot}(\psi)) \,. \tag{19}$$

What about the matrix representations of Rot_ψ in another base, say $\vec{e}\,'$? The answer to that question is made a lot easier than it seems by the availability of the change of basis matrix $P^{\boldsymbol{e}}_{\boldsymbol{e}'} = P$. Indeed, we can use the matrix P and its inverse to go back and forth between the coordinates in the basis $\vec{e}$ and $\vec{e}\,'$. In particular, we have

$$\boldsymbol{v}'_\psi = P\boldsymbol{v}_\psi = PR_\psi\boldsymbol{v} = \underbrace{PR_\psi P^{-1}}_{R'_\psi}\boldsymbol{v}' \,. \tag{20}$$

Stated otherwise, the matrix representation of $\mathrm{Rot}(\psi)$ in the canonical basis $\vec{e}\,'$ is given by the following recipe: $R'_\psi = \mathrm{Mat}_{\vec{e}\,'}(\mathrm{Rot}(\psi)) = PR_\psi P^{-1}$. Thus, we confirm that different basis yields different matrix representations of the same linear application. Now, it is worth mentioning that not all representations are equal: we usually favor "special" representations, e.g. matrices that are parsimonious with respect to the number of non-zero coefficients, because these matrices are more convenient to interpret and manipulate. As a consequence, a recurrent theme of linear algebra is to find bases for which the representation of a linear application is "special".

3 Special matrices

Here follow a few examples of special matrices.

3.1 Elementary matrix

Elementary matrices are matrices for which all entries are zero except one that has unit value. Elementary matrix can be labelled by the indices of that unit entry, e.g. E_{21} is the elementary matrix that has a one in the second row and the first column. When used to right-multiply vectors, E_{21} return the vector that is zero everywhere except on the second row where it takes value of the first component of the original vector:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ v_1 \\ 0 \end{bmatrix} . \tag{21}$$

When used to left-multiply vectors, E_{21} returns the vector that is zero everywhere except on the first row where it takes value of the second component of the original vector:

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} v_2 & 0 & 0 \end{bmatrix} . \tag{22}$$

These observations generalize to matrix multiplication. When used to right-multiply matrices, E_{21} returns the matrix that is zero everywhere except on the second row where it takes value of the first row of the original vector:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{r}_1 \\ \mathbf{0} \end{bmatrix} . \tag{23}$$

When used to left-multiply matrices, E_{21} returns the vector that is zero everywhere except on the first column where it takes value of the second column of the original vector:

$$\begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_2 & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{24}$$

Every matrix can be seen as linear combinations of elementary matrices. Moreover, elementary matrices can be combined to generate combinatorial operation such as switching rows or column, and more generally to generate any permutations of entries.

3.2 Diagonal matrix

Diagonal matrices only have nonzero entries on the diagonal. The identity matrix I is a diagonal matrix with all diagonal entry equal to one such that for all $\boldsymbol{v}$, we have $I\boldsymbol{v} = \boldsymbol{v}I = \boldsymbol{v}$. When used to right-multiply matrices, diagonal matrices multiple rows by the corresponding diagonal coefficients:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \boldsymbol{r}_1 \\ \lambda_2 \boldsymbol{r}_2 \\ \lambda_3 \boldsymbol{r}_3 \end{bmatrix} \tag{25}$$

When used to left-multiply matrices, diagonal matrices multiple columns by the corresponding diagonal coefficients:

$$\begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \boldsymbol{c}_1 & \lambda_2 \boldsymbol{c}_2 & \lambda_3 \boldsymbol{c}_3 \end{bmatrix} \tag{26}$$

Linear applications that are represented by a diagonal matrix in a basis $\vec{e}$ have a very simple geometrical interpretation. These applications are scaling operation with potentially different scaling along each basis vectors $\vec{e}_i$ indicated by the diagonal entries λ_i. It turns out that linear applications in finite dimensional space over the field of complex numbers are generally diagonalizable. This means that we can find a basis for which the matrix representation of the linear application is diagonal with potentially complex entries. Equivalently, this means that matrices can generally be reduced to complex-valued diagonal matrices.

3.3 Orthogonal matrix

Orthogonal matrices represent another set of linear applications that can easily be interpreted geometrically. Orthogonal matrices have the property that their columns form an orthonormal systems in the canonical basis. This means that the inner product of distinct column is zero while the length of each column vector is one. This is conveniently summarized by writing

$$O^T O = \begin{bmatrix} \boldsymbol{c}_1^T \\ \boldsymbol{c}_2^T \\ \boldsymbol{c}_3^T \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} \boldsymbol{c}_1^T \boldsymbol{c}_1 & \boldsymbol{c}_1^T \boldsymbol{c}_2 & \boldsymbol{c}_1^T \boldsymbol{c}_3 \\ \boldsymbol{c}_2^T \boldsymbol{c}_1 & \boldsymbol{c}_2^T \boldsymbol{c}_2 & \boldsymbol{c}_2^T \boldsymbol{c}_3 \\ \boldsymbol{c}_3^T \boldsymbol{c}_1 & \boldsymbol{c}_3^T \boldsymbol{c}_2 & \boldsymbol{c}_3^T \boldsymbol{c}_3 \end{bmatrix} \tag{28}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

Thus, we have $O^T O = I$, which is equivalent to saying that orthogonal matrices are invertible with inverse $O^{-1} = O^T$. The geometric interpretation is that orthogonal matrices represents linear application that send the canonical basis, which is an orthonormal basis of vectors, to another orthonormal basis of vectors. This means that orthogonal matrices represent rigid transformations that are generalizations of rotations[4].

[4]Orthogonal matrices represent applications that are composition of rotation and reflection operations.