Dynamical Systems

System of ordinary differential equations

$$\dot{X} = F(X) \qquad X = (x_1, \dots, x_n) \in \mathbb{R}^n \qquad \left(\dot{X} = \frac{dX}{dt}\right)$$

Classical result about existence and uniqueness of solution such that $X(0) = x_0$ on some open interval $I \ni x_0$. Picard Lindelöf theorem.

Cond : F is K-Lipschitz

Here : F will be assumed smooth.

Flows and Orbits

The flow is a function $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ parametrized by t such that $X_t \doteq \phi_t(X_0)$ is solution of the system

$$\left.\frac{d\phi_t(X_0)}{dt}\right|_{t=\tau} = F(\phi_\tau(X_0)), \qquad \phi_0(X_0) = X_0$$

$X \in \mathbb{R}^n$: orbit of X : $\underset{\substack{\wedge \\ I \text{ maximal}}}{t} \mapsto \phi_t(X)$

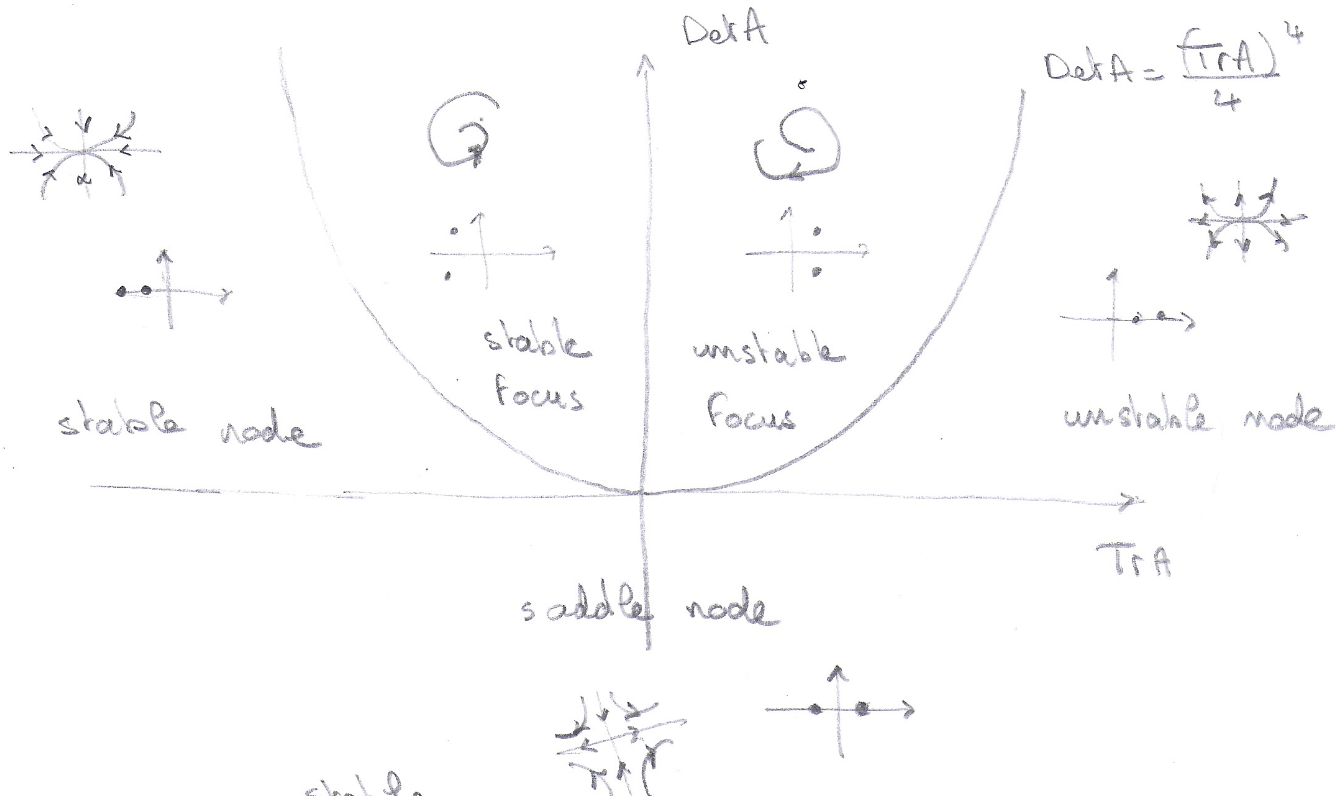
Linear Flows

$$\dot{X} = AX \quad , \quad A \in M_n(\mathbb{R})$$

$$\phi_t = e^{At} = \sum_{n \geq 0} \frac{A^n t^n}{n!}$$

Special orbit: $\{0\}$ → equilibrium / stationary point

Phase portrait of 2D linear flows



$$\mathbb{R}^N = \underset{\substack{\uparrow \\ \text{unstable}}}{E_u} + \overset{\substack{\text{stable} \\ \downarrow}}{E_s} + \underset{\substack{\uparrow \\ \text{central}}}{E_c}$$

$n_+(A) = \dim E_u$

$n_-(A) = \dim E_s$

$n_0(A) = \dim E_c$

Invariant spaces

Topological equivalence

(1) $\dot{X} = F(X)$, $X \in \mathbb{R}^N$, (2) $\dot{Y} = G(Y)$, $Y \in \mathbb{R}^N$

(1) and (2) are topologically equivalent if there is a homeomorphism $H : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\begin{tikzcd} \mathbb{R}^N \arrow[r, "\phi_t"] \arrow[d, shift left, "H"] & \mathbb{R}^N \arrow[d, shift left, "H"] \\ \mathbb{R}^N \arrow[r, "\psi_t"] \arrow[u, shift left, "H^{-1}"] & \mathbb{R}^N \arrow[u, shift left, "H^{-1}"] \end{tikzcd}$$

ϕ flow of (1)

ψ flow of (2)

$H \circ \phi_t = \psi_t \circ H$

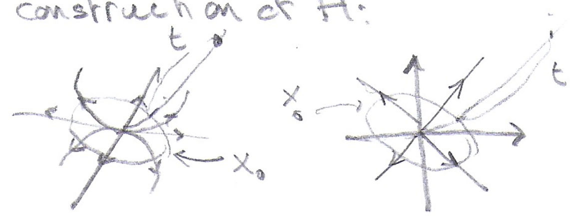
Prop:

Linear flows associated to A and B, for A and B with $n_0(A) = n_0(B) = 0$, are topologically equivalent if $n_+(A) = n_+(B)$ where $n_+(A)$: # eigenvalues of A $(n_-(A) = n - n_+(A))$

Remark: if H is diffeomorphism, linear flows associated to A and B, for A and B with $n_0(A) = n_0(B) = 0$, are differentially equivalent if A and B are similar (identical eigen values). Proof?

Idea of proof:

* enough to show result for $n_+(A) = n$
* lemma: existence of positive bilinear form r such that $D_A r \geq 0$.
* construction of H: $H(\phi_t X_0) = \psi_t X_0$



Bifurcation

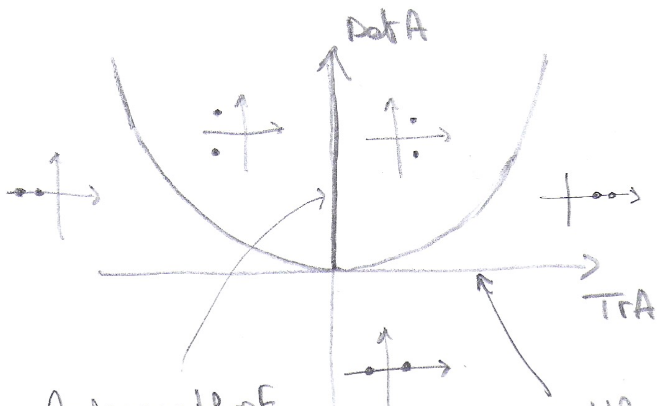
Parametrized dynamical system : $\dot{X} = F(X, \lambda)$, $X \in \mathbb{R}^n$, $\lambda \in \Lambda$

Parameter $\lambda_0 \in \Lambda$ is regular if there is an open neighborhood U such that any system $\dot{X} = F(X, \lambda)$, $\lambda \in U$ is topologically equivalent to $\dot{X} = F(X, \lambda_0)$.

Λ_0 : set of regular values

$\Lambda_b = \Lambda \setminus \Lambda_0$: bifurcation set.

At a bifurcation point : $\dot{X} = F_b(X, \lambda_b)$ is not structurally stable : there is a perturbation P such that $\forall \delta$ $\exists \varepsilon < \delta$ $\dot{X} = F(X) + \varepsilon P(X)$ is not topologically equivalent to $\dot{X} = F(X)$.



Andronov Hopf bifurcation : conjugate eigenvalues cross the imaginary axis

saddle-node bifurcation : eigenvalue become zero

Local analysis of hyperbolic equilibria

Nonlinear system: $\dot{X} = F(X)$. Equilibrium: $F(0) = 0$

Linearization: $\dot{X} = LX$, $L = D_X F|_{X=0}$

Hyperbolic equilibria: no eigenvalue with zero real part

Non-hyperbolic equilibria: L has an eigenvalue with zero real part.

Hartman Grobman Theorem

A locally hyperbolic dynamical system $\dot{X} = F(X)$ is locally topologically equivalent to its linearization.

Idea of proof: (Perko)

$*$ $L = \left[\begin{array}{c|c} N & 0 \\ \hline 0 & P \end{array}\right]$, $\phi_t(X_0) = \begin{bmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{bmatrix}$, y on unstable space, z on stable space

$*$ $\left.\begin{array}{l} \tilde{y}(y_0, z_0) = y(1, y_0, z_0) - e^N y_0 \\ \tilde{z}(y_0, z_0) = z(1, y_0, z_0) - e^P z_0 \end{array}\right)$ deviation from linearization

$\hat{y} = \tilde{y}$, $\hat{z} = \tilde{z}$ on small enough ball around 0 and 0 otherwise.

$*$ $A = e^N$, $B = e^P$; proper normalization $\rightarrow$ $\|A\| < 1$, $\|B^{-1}\| < 1$

$\underset{\substack{\uparrow \\ \text{linear flow}}}{\Psi_1} X = e^L X = \begin{bmatrix} Ay \\ Bz \end{bmatrix}$, consider the map: $\phi_1(X) = \begin{bmatrix} Ay + \hat{y}(y,z) \\ Bz + \hat{z}(y,z) \end{bmatrix}$

Lemma: There exists an homeomorphism H such that $\hat{H} \circ \phi_1 = \Psi_1 \circ \hat{H}$

$\hat{H}(x) = \begin{bmatrix} U(y,z) \\ V(y,z) \end{bmatrix}$, $\hat{H} \circ \phi_1 = \Psi_1 \circ \hat{H}$ implies

$$A U'(y,z) = U\left(Ay + \hat{Y}(y,z), Bz + \hat{Z}(y,z)\right)$$

the solution ψ can be found by an iterative argument

$$U_0(y,z) = y, \quad U_{k+1}(y,z) = A^{-1} U_k\left(Ay + \hat{Y}(y,z), Bz + \hat{Z}(y,z)\right)$$

which can be shown to be a Cauchy sequence

* Define $H = \int_0^1 \Psi_{-s} \hat{H} \phi_s \, ds$, $T = \phi_1$, $L = \Psi_1$

$$\Psi_t \circ H = \left(\int_0^1 \Psi_{t-s} \hat{H} \phi_{s-t} \, ds\right) \phi_t = \left(\int_{-t}^{1-t} \Psi_{-s} \hat{H} \phi_s \, ds\right) \phi_t$$

$$= \left[\int_{-t}^{0} \Psi_{-s} \hat{H} \phi_s \, ds + \int_0^{1-t} \Psi_{-s} \hat{H} \phi_s \, ds\right] \phi_t$$

$$= \left(\int_0^1 \Psi_{-s} \hat{H} \phi_s \, ds\right) \phi_t = H \circ \phi_t$$

$$\int_{-t}^{0} \Psi_{-s} \hat{H} \phi_s \, ds = \int_{-t}^{0} \Psi_{-s-1} \Psi_1 \hat{H} \phi_s \, ds$$

$$= \int_{-t}^{0} \Psi_{-s-1} \hat{H} \phi_1 \phi_s \, ds$$

$$= \int_{-t}^{0} \Psi_{-(s+1)} \hat{H} \phi_{(s+1)} \, ds$$

$$= \int_{1-t}^{1} \Psi_s \hat{H} \phi_s \, ds$$