Local analysis of nonhyperbolic equilibria

Local bifurcation happens when equilibria becomes nonhyperbolic: $L_{\lambda_0} = D_x F|_{\lambda=\lambda_0}$ has an eigenvalue with zero real part.

E_c: central space

$E_h = E_u + E_s$: hyperbolic space

Center manifold theorem

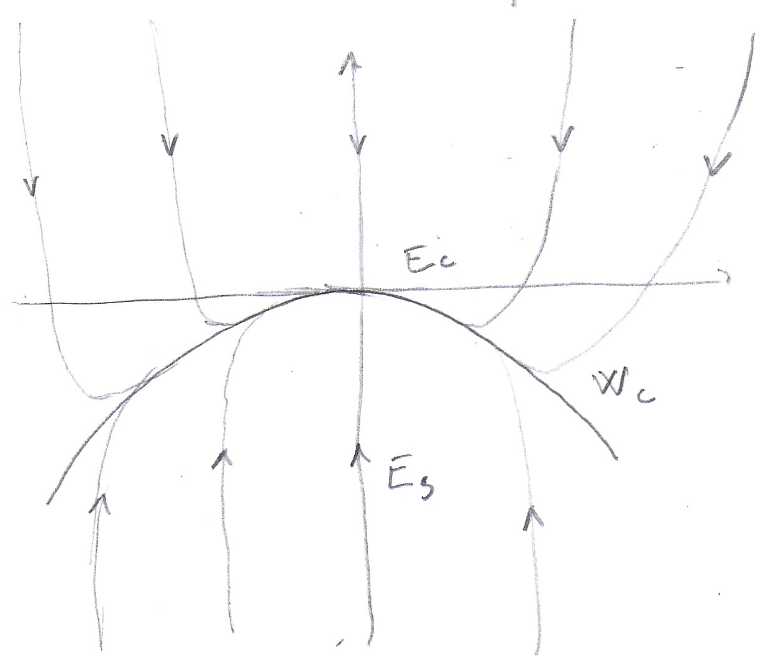
There exists a nonlinear mapping

$$H : E^c \to E^h, \quad H(0) = 0, \quad DH(0) = 0$$

and a neighborhood U of $x=0$ in $\mathbb{R}^n$ such that the manifold $M = \{x + H(x) \mid x \in E^c\}$, the center manifold, has the following properties:

* Invariance: if $x(0) \subset M \cap U$, then $x(t) \in M$ as long as $x(t) \in U$.

* Attractivity: if $E^u = \phi$, all solutions in U tends exponentially to some solution in M

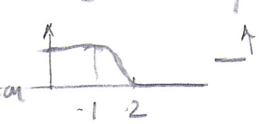


Idea of proof for global CMT (Vanderbauwhede)

* $\dot{X} = LX + G(X)$, $\quad G(X) = F(X) - LX$, $\quad G(0) = 0$, $L = DF(0)$

reduction to compact perturbation: $G_\varepsilon(X) = \rho\left(\frac{|X|}{\varepsilon}\right) G(X)$
with $\|G_\varepsilon\|_{C_1} = O(1)\,\varepsilon$.

ρ: even smooth function



* space of slow-growth functions: $0 < \eta < \beta =$ spectral gap of L.

$S_\eta = \{ Y : \mathbb{R} \to \mathbb{R}^N \mid \|Y\|_\eta = \sup_t e^{-\eta|t|} |Y(t)| < \infty \} \to$ Banach space

* Solution under "variation of constant formula"

$$Y(t) = e^{L(t-t_0)} Y(t_0) + \int_{t_0}^{t} e^{L(t-s)} G(Y(s))\, ds$$

* Projection on stable, unstable, center space

$$\begin{aligned} Y(t) = {} & \Pi_c \left[e^{L(t-t_c)} Y(t_c) + \int_{t_c}^{t} e^{L(t-s)} G(Y(s))\, ds \right] \\ & + \Pi_s \left[e^{L(t-t_s)} Y(t_s) + \int_{t_s}^{t} e^{L(t-s)} G(Y(s))\, ds \right] \\ & + \Pi_u \left[e^{L(t-t_u)} Y(t_u) + \int_{t_u}^{t} e^{L(t-s)} G(Y(s))\, ds \right] \end{aligned}$$

Take $t_c = 0$, $\Pi_c(Y(0)) = X_c \in E_c$

$$\begin{aligned} Y(t) = {} & e^{Lt} X_c + \int_0^t e^{L(t-s)} \Pi_c G(Y(s))\, ds \\ & + \int_{-\infty}^{t} e^{L(t-s)} \Pi_s G(Y(s))\, ds + \int_{t}^{+\infty} e^{L(t-s)} \Pi_u G(Y(s))\, ds = \Gamma_{X_c}[Y](t) \end{aligned}$$

* The map $\Gamma_{X_c} : Y_\eta \to Y_\eta$ is a contraction. Fixed point trajectory: Y_{X_c}. The (global) center manifold is defined as

$M = \{ Y_{X_c}(0) \mid X_c \in E_c \}$, $\quad H(X_c) = Y_{X_c}(0) - X_c$.

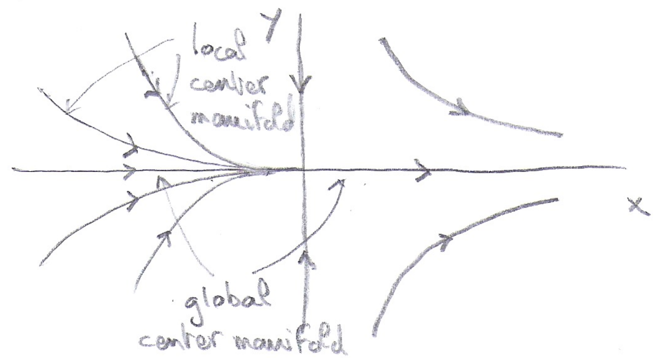
* Properties of M: tangency to E^c, invariance by the flow ϕ, exponential attractivity: if $x(t) \xrightarrow[t\to+\infty]{} 0$, then there is $x_c \in E^c$, $e^{\eta t}|x(t) - Y_{x_c}(t)| \xrightarrow[t\to+\infty]{} 0$

Local center manifold vs. Global manifold

$\dot{x} = x^2$, $\dot{y} = -y$

vector fields C^{k+1}

$\Rightarrow$ H of class C^k



Center manifold reduction

$$X(t) = X_c(t) + H(X_c(t)) \quad , \quad X(0) \in M \cap U, \quad X_c \in E_c$$

$$\dot{X}(t) = \dot{X}_c(t) + DH(X_c(t))\,\dot{X}_c(t) = F(X_c(t) + H(X_c(t)))$$

Projection to E_c: $\dot{X}_c = \Pi_c F(X_c + H(X_c)) = f(X_c) \rightarrow$ ODE for $X_c(t)$

Projection to E_h: $DH(X_c) F(X_c) = \Pi_h F(X_c + H(X_c)) \rightarrow$ determine H

Useful result to simplify the dynamics in the vicinity of an equilibrium by considering the dynamics in the central space E_c.

Computational interest of center manifold theorem

1) Convert $\dot{X} = F(X)$ in "diagonal" form
2) Use a series expansion for H.
3) Determine series coefficients using CM reduction
4) Substitute the approximate expression for H to determine the flow.

Ex: $\dot{x} = xy$, $\dot{y} = -y - x^2$, one zero eigenvalue $\dim E_c = 1$

$h(x) = ax^2 + bx^3 + \dots$, $h'(x) = 2ax + 3bx^2 + \dots$

$x \in E_c$: $\dot{x} = x h(x)$, $\dot{y} = h(x)\dot{x} = -h(x) - x^2$

$\rightarrow (2ax + 3bx^2 + \dots) \, x \, (ax^2 + bx^3 + \dots) = -(ax^2 + bx^3 + \dots) - x^2$

which implies: $a = -1$, $b = 0$

$\hookrightarrow$ on the center space: $\dot{x} = x(-x^2 + O(x^4)) = -x^3 + O(x^5)$

In general: $\dot{x} = Cx + F_c(X, Y) \leftarrow E_c$, C zero eigenvalues

$\dot{y} = Py + F_s(X, Y) \leftarrow E_s$, S positive real part eigenvalues

$F_c(0,0) = F_s(0,0) = 0$, $DF_c = DF_s = 0$

$H : E_c \rightarrow E_s$ satisfies:

* $\dot{X} = CX + F_c(X, H(X))$
* $\dot{Y} = DH(X)\left[CX + F_c(X, H(X))\right] = PH(X) + F_s(X, H(X))$.

Categorization of bifurcations

Extended CM theorem:

$$\dot{x} = Cx + F_c(x,y,\mu), \qquad F_c(0,0,\overset{\mu}{0}) = F_s(0,0,\overset{\mu}{0}) = 0 \quad \text{(bifurcation parameter)}$$

$$\dot{y} = Py + F_s(x,y,\mu) \qquad DF_c = DF_s = 0$$

$$\dot{\mu} = 0$$

$\Rightarrow$ the center manifold can be parametrized as $y = H(x,\mu)$

If the center manifold is 1-dimensional, center manifold reduction yields a 1-dimensional ODE

$$\dot{x} = f(x,\mu) \text{ with } f(0,0) = 0, \quad \partial_x f(0,0) = 0$$

Taylor expansion of f:

$$\dot{x} = \cancel{f(0,0)} + \cancel{\partial_x f(0,0)}\, x + \partial_\mu f(0,0)\,\mu$$
$$+ \frac{1}{2}\left(\partial_{xx} f(0,0)\, x^2 + 2\partial_{x\mu} f(0,0)\, x\mu + \partial_{\mu\mu} f(0,0)\,\mu^2\right)$$
$$+ O(|x^3|, |\mu|^3).$$

Graphical analysis of equilibria stability shows that only 3 types of bifurcation are possible:

$\dot{x} = \mu - x^2$, saddle-node; $\dot{x} = \mu x - x^2$, transcritical; $\dot{x} = \mu x \pm x^3$, pitchfork

$\rightarrow$ Exercise!

$\boxed{\text{Normal Forms}}$

Normal form theory

* Categorization of bifurcation theory via center manifold reduction becomes unwieldy when $\dim E_c > 1$.

* Idea of normal form theory: identify canonical dynamical systems by finding polynomial change of variables which locally improves (simplify) the nonlinear system.

Poincaré - Dulac theorem

$$\dot{X} = LX + G(X), \quad G(0)=0, \quad DG(0)=0, \quad L \text{ diagonal}$$

If the eigenvalues of L are non-resonant up to order n then there is polynomial change of variable close to identity:

$$x_1 = y_1 + p_1(y_1, \dots, y_n)$$
$$\vdots$$
$$x_n = y_n + p_n(y_1, \dots, y_n)$$

for which the system is reducible to the form

$$\dot{Y} = LY + O(|Y|^{n+1})$$

Resonance: A resonance relation among $\lambda_1, \dots, \lambda_n$ exists if there are $m_1, \dots, m_n \in \mathbb{N}^n$, $\sum_j m_j \geq 2$ such that for some r: $\lambda_r = \sum_j m_j \lambda_j$. order $= \sum_j |m_j|$

Example: $\lambda_1 = \lambda_2 + \lambda_3$ order 2, $\cancel{2\lambda_1 = 3\lambda_2}$,

$\lambda_1 = -\lambda_2 \Rightarrow \lambda_s = \lambda_s + k(\lambda_1 + \lambda_2)$, $k = 1, 2, 3, \dots$

Idea of proof:

* Taylor expansion of F:

$$\dot{X} = LX + \underbrace{P_2(X) + \cdots + P_r(X)}_{P(X)} + O(|X|^{r+1})$$

homogeneous vector polynomial of degree r

* Taylor expansion of change of variable:

$$X = Y + Q(Y) = Y + Q_2(Y) + \cdots + Q_r(Y)$$

* Recurrence on the order r

$$\dot{X} = LX + P_r(X) + O(|X|^{r+1})$$

Change of variable: $X = Y + Q_r(Y)$

We look for Q_r such that $\dot{Y} = P_{r+1}(Y) + O(|Y|^{r+2})$

* By substitution:

$$\dot{X} = (I + D_Y Q_r)\dot{Y} = L(Y + Q_r(Y)) + P_r(Y + Q_r(Y)) + O(|Y|^{r+1})$$

Since $\dot{Y} = LY + o(Y)$, we have

$$\dot{Y} + \underbrace{D_Y Q_r LY}_{\text{degree } r} = LY + \underbrace{LQ_r(Y)}_{\text{degree } r} + \underbrace{P_r(Y)}_{\text{degree } r} + O(|Y|^{r+1})$$

Desired form if: $D_Y Q_r LY - LQ_r(Y) = P_r(Y)$

$$\overset{\shortparallel}{Ad_L[Q_r]} : H_r \longrightarrow H_r$$

linear map stabilizing the space of homogeneous polynomial of a given degree

$Ad_L[Q] = P$: homological equation

* <u>lemma</u> : $Ad : H_r \to H_r$ is invertible if there is no resonance of order r.

<u>proof</u> : Consider the basis of eigenvector $(e_1, \dots, e_n)$ associated to $(\lambda_1, \dots, \lambda_n)$, assumed distinct for simplicity.

A basis of H_r is given by

$$Q_{i,\underline{m}} = y_1^{m_1} \cdots y_n^{m_n} e_i \ , \quad m_1 + \dots + m_n = r$$

We have : $Ad_L [Q_{i,\underline{m}}] = \underbrace{(m_1\lambda_1 + \dots + m_n\lambda_n - \lambda_i)}_{\neq 0 \text{ in the absence of resonance}} Q_{i,\underline{m}}$