Resonant monomial

* In the presence of resonance, e.g., $\lambda_i = m_1\lambda_1 + \dots + m_n\lambda_n$ the monomial term $x_1^{m_1} \dots x_n^{m_n} e_i$ cannot be cancelled via polynomial change of variable: resonant monomial

* Adapting the procedure of the Poincaré-Dulac theorem we expect to be able to reduce a dynamical system to a form:

$$\dot{X} = LX + P_n(X) + O(|X|^{n+1})$$

where $P_n(X)$ is the sum of resonant monomials up to order n.

* Such a form only depends on the linear part $L = DF(0)$ (considered under its Jordan form) and is called a normal form.

Normal Form theorem

$\dot{X} = F(X)$, F of class C^k, $F(0) = 0$, $DF(0) = L$

Choose a complement G_k such that $G_k + \operatorname{ad}_L H_k = H_k$.

There is an analytical change of coordinates such that in a neighborhood of the origin the system $\dot{X} = F(X)$ is transformed into:

$$\dot{Y} = LY + Q_2(Y) + \dots Q_r(Y) + o(|Y|^r), \quad Q_i \in G_i, \quad 2 \le i \le r$$

Normal form for $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow \begin{array}{l} \dot{x} = -y + o(|x|,|y|) \\ \dot{y} = x + o(|x|,|y|) \end{array}$

* $Ad_L : H_2 \to H_2$:

$$Ad_L \begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ x^2 \end{pmatrix} - \begin{pmatrix} -2xy \\ 0 \end{pmatrix} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$$

$\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = x^2 e_x$. basis: $x^2 e_x \quad xy\, e_x \quad y^2 e_x$

$x^2 e_y \quad xy\, e_y \quad y^2 e_y$

Matrix:

$$\begin{array}{cc} & \text{basis} \\ \text{basis} & \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array} \to \text{invertible: all quadratic terms can be removed}$$

* Evaluating the matrix of $Ad_L : H_3 \to H_3$ reveals that the resonant terms $(x^2+y^2)(x e_x + y e_y)$ and $(x^2+y^2)(-y e_x + x e_y)$ cannot be removed.

Thus the system can be transformed to:

$$\begin{aligned} \dot{x} &= -y + (x^2+y^2)(ax - by) + o(|x,y|^3) \\ \dot{y} &= x + (x^2+y^2)(ay + bx) + o(|x,y|^3) \end{aligned}$$

Normal form for Hopf bifurcation

* Normal form for $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in polar coordinate

$$\left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right. \Rightarrow \left| \begin{array}{l} \dot{r} = ar^3 + o(|r|^3) \\ \dot{\theta} = 1 + br^2 + o(|r|^3) \end{array} \right.$$

$r = 0$ stable if $a > 0$.

Hopf bifurcation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x,y;\mu) = L_\mu \begin{pmatrix} x \\ y \end{pmatrix} + G(x,y;\mu), \quad \mu = 0$$

$F(0,0;0) = 0$, $DF(0,0;\mu) = L_\mu$, $L_0 = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

$G(0,0;0) = 0$, $DG(0,0;0) = 0$

From normal form analysis; the above system can be reduced to $G(x,y;0) = (x^2+y^2)\begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}$

Canonical model (normal form for Hopf bifurcation):

$$\dot{x} = \mu x - \omega y + (x^2+y^2)(ax - by) + o(|x,y|^3)$$
$$\dot{y} = \omega x + \mu y + (x^2+y^2)(ay + bx) + o(|x,y|^3)$$

In complex notations: $z = x + iy$

$$\boxed{\dot{z} = (\mu + i\omega) z + (a - ib)|z|^2 z} + o(|z|^3)$$

Truncated normal form is a topological normal form: no need for higher terms;

Codimension of bifurcation

* The notion of codimension is related to the topological notion of transversality (not discussed).

* Informaly, the codimension of a bifurcation refers to the number of conditions (equalities on the bifurcation parameters) that need to be met at the bifurcation points:

* Local bifurcation occurs at nonhyperbolic equilibria for which the linearization L of a system $\dot{X} = F(X)$ has eigenvalue with zero real parts. The codimension is the number of eigenvalues with zero real part (conjugate eigenvalues count for one)

* Codimension 1 bifurcation corresponds to

$$L = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right] \qquad \text{or} \qquad L = \left[\begin{array}{cc|c} 0 & -\omega & 0 \\ \omega & 0 & \\ \hline \multicolumn{2}{c|}{0} & A \end{array}\right]$$

saddle - node = real eigenvalue crosses the imaginary axis

Hopf bifurcation = complex conjugate eigenvalue cross the imaginary axis

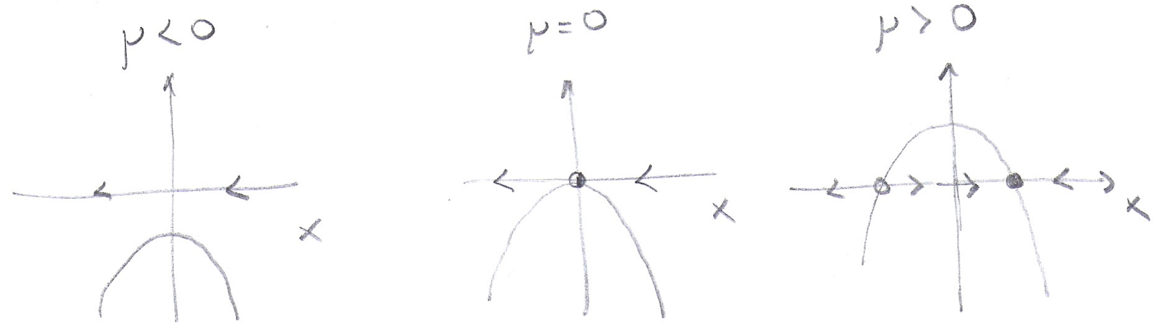
* Codimension 2:

$$L = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \\ \hline 0 & A \end{array}\right], \quad \left[\begin{array}{ccc|c} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & \\ 0 & 0 & 0 & \\ \hline \multicolumn{3}{c|}{0} & A \end{array}\right], \quad \left[\begin{array}{cccc|c} 0 & -\omega_1 & 0 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \omega_2 & \\ 0 & 0 & -\omega_2 & 0 & \\ \hline \multicolumn{4}{c|}{0} & A \end{array}\right]$$

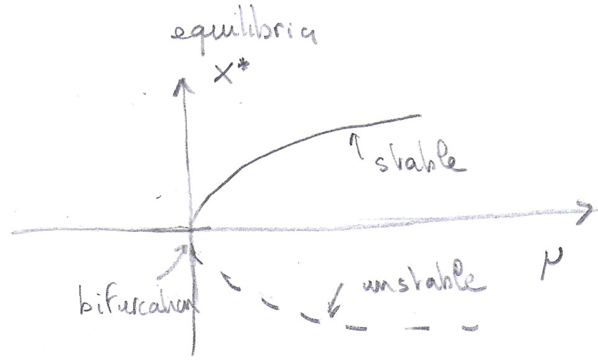
Codimension-1 local bifurcation

* saddle-node (fold) bifurcation

↳ "proper" saddle node : normal form $\dot{x} = \mu - x^2$



bifurcation diagram

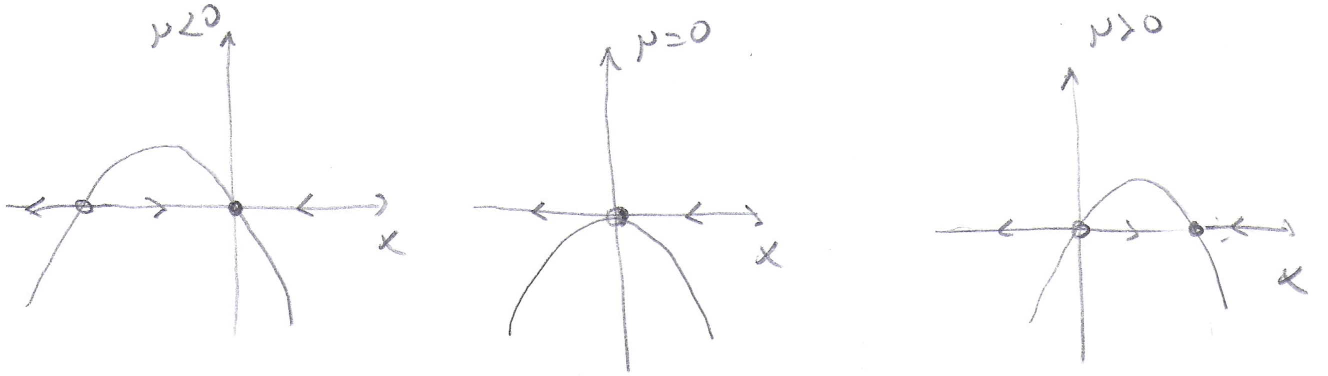


$\dot{X} = F(X, \mu)$ with $F(X_0, \mu_0) = 0$ is locally topologically equivalent to $\dot{x} = \mu - x^2$ if

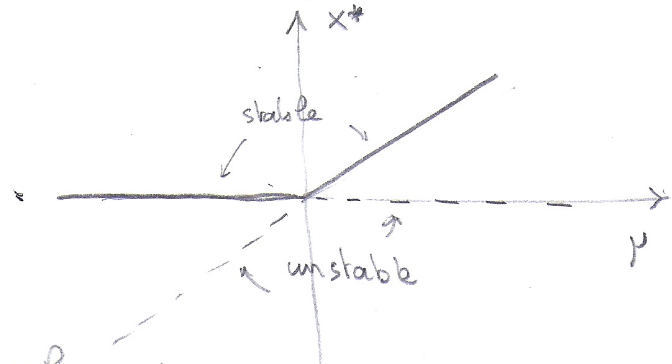
- nonhyperbolicity: $D_{X_0} f(x_0, \mu_0)$ has a simple zero eigenvalue with right eigenvector: V
  left eigenvector: U
- transversality: $U . \partial_{\mu_0} f(X_0, \mu_0) \neq 0$: the eigenvalue crosses the imaginary axis when μ crosses μ_0
- non degeneracy: $U . D_x^2 (X_0, \mu_0)(V, V) \neq 0$: the dominant effect is due to quadratic terms

↳ "special" saddle node: different nondegeneracy / transversality conditions

↳ transcritical bifurcation: normal form $\dot{x} = \mu x - x^2$



bifurcation diagram

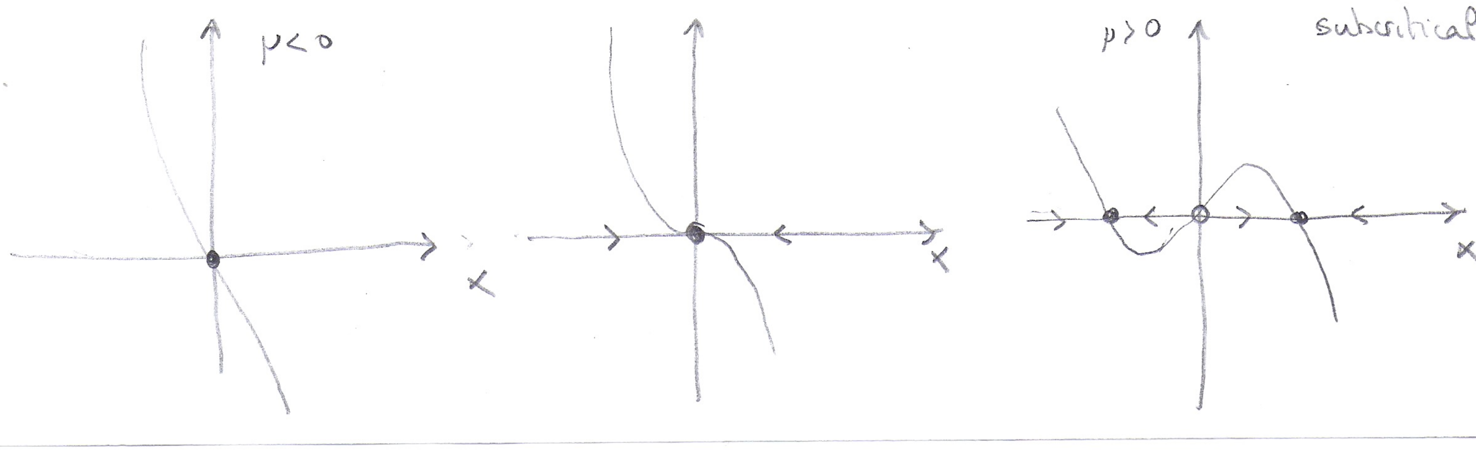


"exchange of stability"

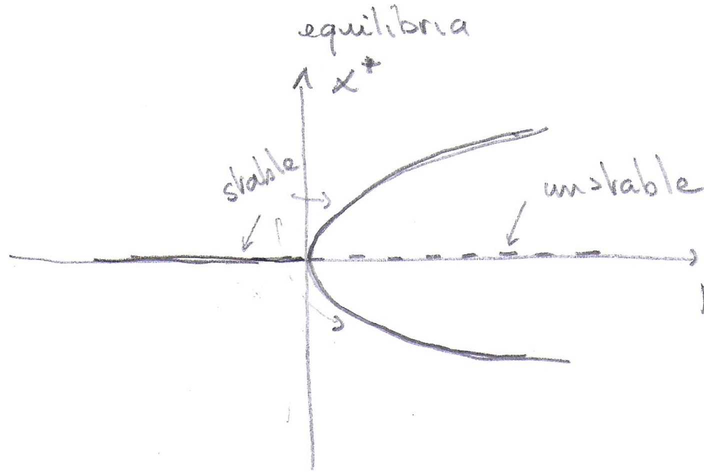
Topological equivalence:

- transversality: under assumption that $\partial_\mu F(x_0,\mu_0) = 0$ we require $U \cdot D_{\mu x} F(x_0,\mu_0)(V) \neq 0$: x_0 remains an equilibrium.

↳ pitchfork bifurcation normal form $\dot{x} = \mu x - x^3$ (supercritical), $\dot{x} = \mu x + x^3$ (subcritical)



"bifurcation" diagram



Topological equivalence: (typically for $F(-x,\mu) = -F(x,\mu)$)

- transversality: under assumption that $\partial_\mu F(x_0,\mu_0) = 0$ we require $U \cdot D_{\mu x} F(x_0,\mu_0)(V) \neq 0$ ∴ x_0 remains an equilibrium

- nondegeneracy: under assumption that $U \cdot D_x^2 (x_0,\mu_0)(V,V) = 0$, we require $U \cdot D_x^3 (x_0,\mu_0)(V,V,V) \neq 0$.

$U \cdot D_x^3 (x_0,\mu_0)(V,V,V) < 0 \rightarrow$ supercritical
$> 0 \rightarrow$ subcritical

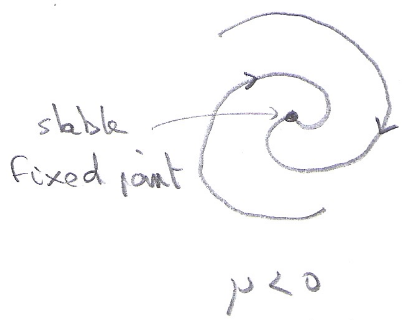
* Andronov-Hopf bifurcation: (2-D center manifold)
↳ normal form in polar coordinates (infinite number of resonance):

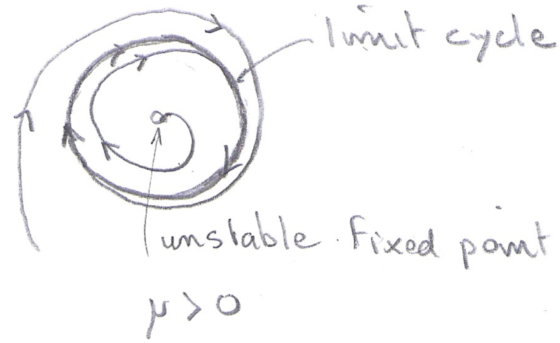
$$\dot{r} = \mu r + a_1 r^3 + a_2 r^5 + a_3 r^7 + \dots$$

$$\dot{\theta} = \omega + b_1 r^2 + b_2 r^4 + b_3 r^6 + \dots$$

↳ supercritical Hopf bifurcation $a_1 < 0$

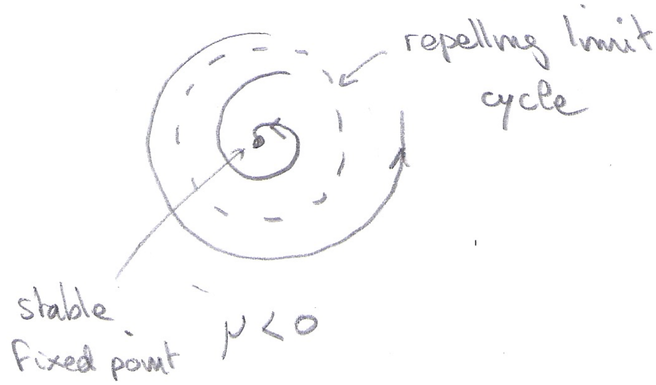
$$\begin{cases} \dot{r} = \mu r - r^3 & \leftarrow \text{stabilizing "}r^3\text{"} \\ \dot{\theta} = \omega + \cancel{b r^2} \end{cases}$$

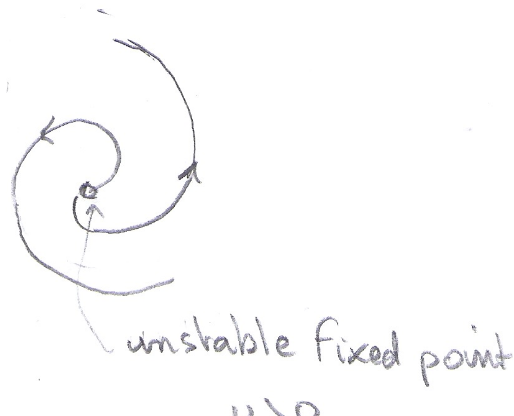




↳ subcritical Hopf bifurcation $a_1 > 0, a_2 < 0$

$$\begin{cases} \dot{r} = \mu r + r^3 & \leftarrow \text{destabilizing "}r^3\text{"} \\ \dot{\theta} = \omega + \cancel{b r^2} \end{cases}$$





Topological equivalence for Hopf bifurcation

$\dot{X} = F(X, \mu)$ with $F(X_0, \mu_0) = 0$ is locally topologically equivalent to $\dot{r} = (\mu + ar^2) r$, $\dot{\theta} = \omega$ if

- nonhyperbolicity: $D_x F(X_0, \mu_0)$ has a simple pair $(\lambda, \bar{\lambda})$ of pure imaginary eigenvalues and no other eigenvalues have zero real part.
- transversality: $\partial_\mu \, Re(\lambda(\mu))|_{\mu=\mu_0} \neq 0$: the pair of eigenvalues cross the imaginary axis when μ crosses μ_0.
- nondegeneracy: complicated except for 2-D flow it boils down to checking that $a_1 \neq 0$. The dominant effect is due to r^3.