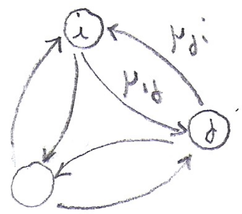
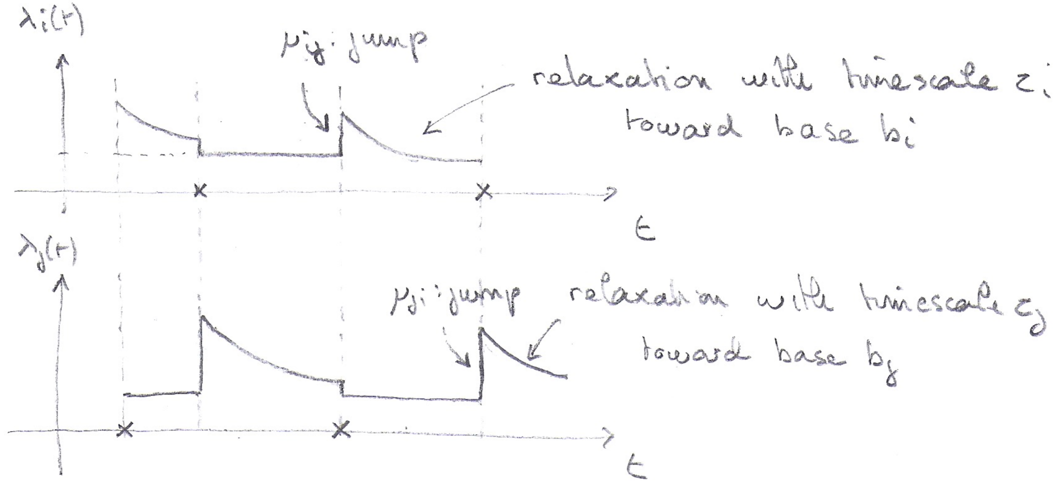
Excitatory heterogeneous intensity-based model



network topology with K neurons



Stochastic equation:

$$\lambda_i(t) = \lambda_i(0) + \underbrace{\frac{1}{\tau_i}\int_0^t (b_i - \lambda_i(s))\,ds}_{\text{relaxation}} + \underbrace{\sum_{j\neq i} \mu_{ij} \int_0^t \overbrace{N_j(ds)}^{\text{stochastic intensity } \lambda_j(t)}}_{\text{interaction}} + \underbrace{\int_0^t (b_i - \lambda_i(s))\, \overbrace{N_i(ds)}^{\text{stochastic intensity } \lambda_i(t)}}_{\text{reset}}$$

Infinitesimal generator: test function $f : (\mathbb{R}^+)^K \to \mathbb{R}$.

$$\mathcal{L}[f](\lambda_1, \dots, \lambda_K) = \sum_i \frac{b_i - \lambda_i}{\tau}\, \partial_{\lambda_i} f(\lambda_1, \dots, \lambda_K)$$

$$+ \sum_i \left[f(\lambda_1 + \mu_{1i}, \dots, b_i, \dots, \lambda_K + \mu_{Ki}) - f(\lambda_1, \dots, \lambda_K) \right] \lambda_i$$

If interactions are equivalent to a smooth drift $\alpha_i(t)$ then Fokker-Planck equation reads

$$\partial_t p_i(\lambda_i, t) = -\partial_{\lambda_i}\Big[\underbrace{\Big(\frac{b_i - \lambda_i}{\tau_i} + \alpha_i(t)\Big)}_{\text{continuous}} p_i(\lambda_i, t) \Big] - \underbrace{\lambda_i\, p_i(\lambda_i, t)}_{\text{death rate = spiking}} + \underbrace{\int_0^{+\infty} \lambda\, p_i(\lambda, t)\, d\lambda\, \delta_{b_i}(\lambda_i)}_{\text{reset}}$$

Thermodynamic mean-field limit

single neuron $\sim$ population of N neurons with interpopulation connections $\frac{\mu_{ij}}{N}$

a neuron from population i is subjected to

$$\alpha_i^N(t) = \frac{1}{N} \sum_{j \neq i} \sum_{n=1}^{N} \mu_{ij} \, dN_{j,n}(t) \xrightarrow[N \to +\infty]{} \sum_{j \neq i} \mu_{ji} \underbrace{\int_0^{+\infty} \lambda \, p_j(\lambda, t) \, d\lambda}_{\beta_j(t)}$$

$\uparrow$

law of large number

The stationary Fokker Planck equation is solved by

$$p_i(\lambda) = \frac{e^{z_i(\lambda - r_i)}}{|s_i - \lambda|} \left| \frac{s_i - \lambda}{s_i - b_i} \right|^{z_i s_i} \beta_i z_i \, \mathbb{1}_{[b_i, s_i]}(\lambda)$$

Normalization condition $- \int p_i(\lambda) d\lambda = 1 \Rightarrow \beta_i = F(\beta_{j \neq i})$

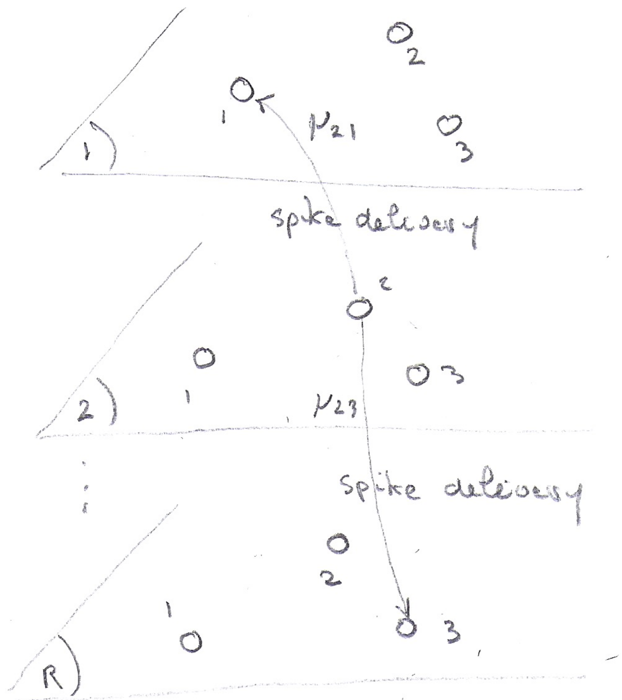
$\uparrow$

input/output relation

Thermodynamic mean-field approximation may fail when the dynamics is correlation dominated or when finite size effects are not negligible.

Replica mean-field limit

Get rid off correlation but keep finite-size effect (adapted to sparse network).

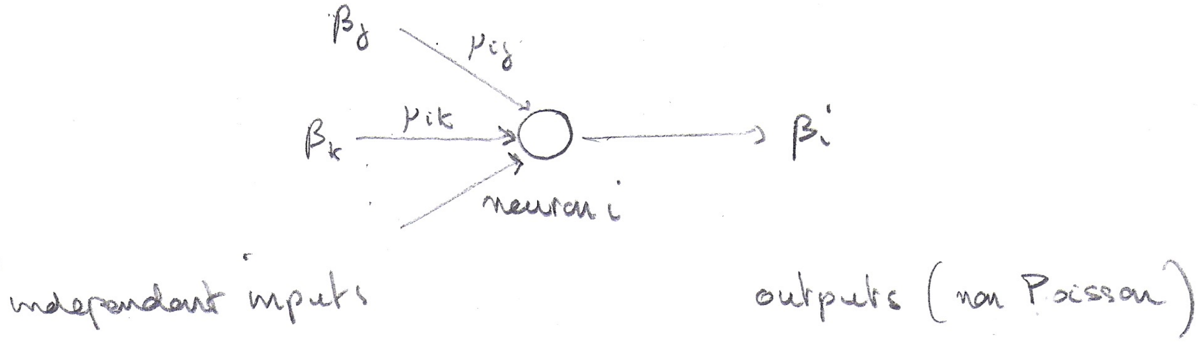


R-replica model are made of R identical copies of K neurons. When a neuron from replica r spikes, it delivers interaction to randomly chosen target neuron across replicas.

The replica-mean-field limit is obtained by taking $R \to +\infty$.

The probability for two replica to interact in finite time vanishes with $R \to +\infty$. Neurons become asymptotically independent.

Self consistency equations :



Generating function formalism

Characterizing stationary state via functional transform, typically the moment-generating function, $u \mapsto \mathbb{E}[e^{u\lambda_i}] = L_i(u)$

Equations specifying L_i are most conveniently obtained via rate conservation principles.

We have:

$$e^{u\lambda_i(t)} = e^{u\lambda_i(0)} + \frac{u}{z_i}\int_0^t (b_i - \lambda_i(s))\, e^{u\lambda_i(s)}\, ds$$

$$+ \sum_{j\neq i} (e^{u\mu_{ij}} - 1)\int_0^t e^{u\lambda_i(s)}\, N_j(ds)$$

$$+ \int_0^t \left(e^{ub_i} - e^{u\lambda_i(s)}\right) N_i(ds)$$

Taking expectation with respect to the stationary measure for which $\mathbb{E}[e^{u\lambda_i(t)}] = \mathbb{E}[e^{u\lambda_i(0)}]$ leads to:

$$\frac{ut}{z_i}\,\mathbb{E}\left[(b_i - \lambda_i)\, e^{u\lambda_i}\right] = \sum_{j\neq i}\left(e^{u\mu_{ij}} - 1\right)\mathbb{E}\left[\int_0^t e^{u\lambda_i(s)}\, N_j(ds)\right]$$

$$+ \int_0^t \mathbb{E}\left[\left(e^{ub_i} - e^{u\lambda_i(s)}\right) N_i(ds)\right]$$

Palm calculus:

$$= \sum_{j\neq i}\left(e^{u\mu_{ij}} - 1\right)\beta_j t\,\mathbb{E}_j^0\left[e^{u\lambda_i(0-)}\right]$$

$$+ \beta_i t\,\mathbb{E}_i^0\left[e^{ub_i} - e^{u\lambda_i(0-)}\right]$$

$$\frac{u}{z_i}\left[b_i\,\mathbb{E}[e^{u\lambda_i}] - \mathbb{E}[\lambda_i e^{u\lambda_i}]\right] = \sum_{j\neq i}\left(e^{u\mu_{ij}}-1\right)\beta_j\,\mathbb{E}^0_j\left[e^{u\lambda_i(0-)}\right]$$

$$+\ \beta_i\left(e^{ub_i} - \mathbb{E}^0_i\left[e^{u\lambda_i(0-)}\right]\right)$$

Papangelou theorem

$$= \sum_{j\neq i}\left(e^{u\mu_{ij}}-1\right)\mathbb{E}\left[\lambda_j e^{u\lambda_i}\right]$$

$$+\ \beta_i e^{ub_i} - \mathbb{E}\left[\lambda_i e^{u\lambda_i}\right]$$

Replica-mean-field

$$= \sum_{j\neq i}\left(e^{u\mu_{ij}}-1\right)\beta_j\,\mathbb{E}\left[e^{u\lambda_i}\right]$$

$$+\ \beta_i e^{ub_i} - \mathbb{E}\left[\lambda_i e^{u\lambda_i}\right]$$

$L_i(u) = \mathbb{E}\left[e^{u\lambda_i}\right]$, $\partial_u L_i(u) = \mathbb{E}\left[\lambda_i e^{u\lambda_i}\right]$;

$$\boxed{\text{System of ODE: } -\left(1+\frac{u}{z_i}\right)\partial_u L_i + \left(\frac{ub_i}{z_i} + \sum_{j\neq i}\left(e^{u\mu_{ij}}-1\right)\beta_j\right)L_i + \beta_i e^{ub_i} = 0}$$

$u=0 \Rightarrow \quad -\partial_u L_i(0) + \beta_i = 0$ ✓

$L_i(u)=1$

The parameters β_i have been introduced as free parameters. Solving the replica-mean-field problem amounts to specifying the parameters β_i.

Reduction to a set of self-consistent equation

In principle, the replica-mean-field ODE admits an infinity of solutions. However "physical solutions", which corresponds to a probabilistic model satisfies strong regularity properties. In particular, they analytical function.

ODE of the form:

$$\begin{cases} -\left(1+\frac{u}{z_i}\right)\partial_u L_i + f_i(u)\, L_i + g_i(u) = 0 \\ f(-z_i) > 0 \end{cases}$$

Simple analytical considerations show that there is a unique continuous (analytic) solution:

$$L_i(u) = \int_{-z_i}^{u} e^{-\int_v^u \frac{f_i(w)}{1+w/z_i}\,dw}\, \frac{g_i(v)}{1+v/z_i}\, dv$$

↑ choice of the bound dictated by continuity

β_i obtained from normalization condition: $L_i(0) = 1$