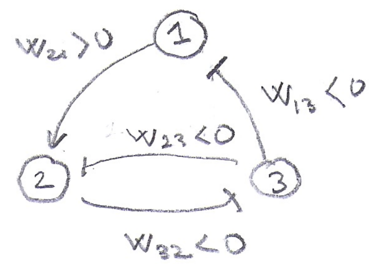
Stochastic neural networks

Example: integrate-and-fire neurons

↳ simplest model: upon spiking, a neuron delivers instantaneous delta Dirac impulses.

Network structure:

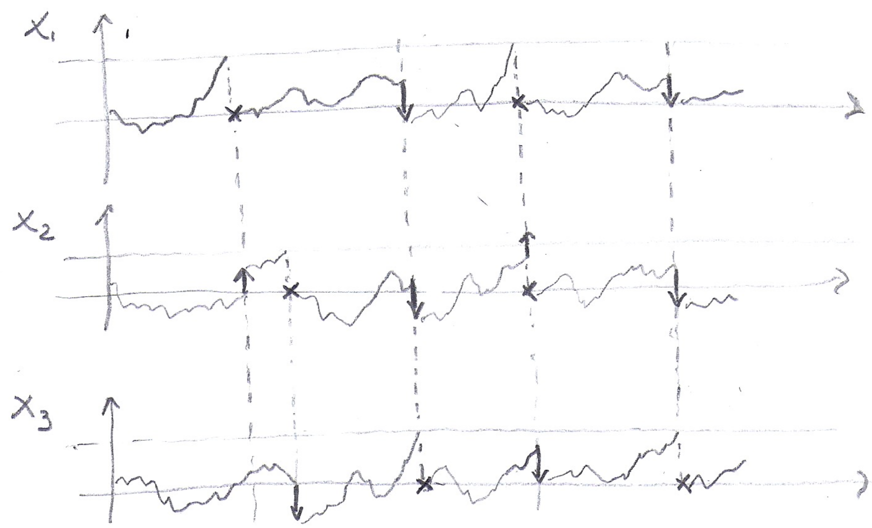


w_{ij}: synaptic weight

Dale's law: neurons come in two flavors: excitatory or inhibitory, i.e., either $w_{ji} \geq 0$ for all j.

or $w_{ji} \leq 0$ for all j.

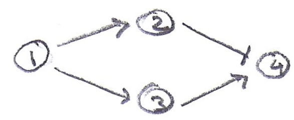
Picture:



⚠ Simultaneous update → well-posed?

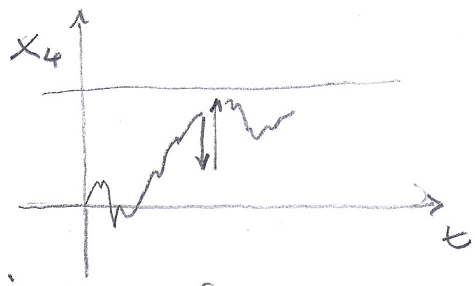
↳ problem with updates ordering.

Problematic case:

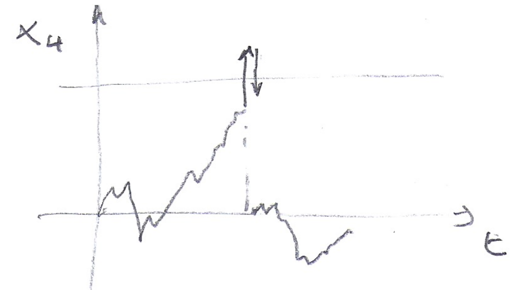


4 simultaneously receives $w_{42} < 0$ and $w_{43} > 0$

2 and 3 may spike simultaneously upon spiking of 1



inhibition first: no spike



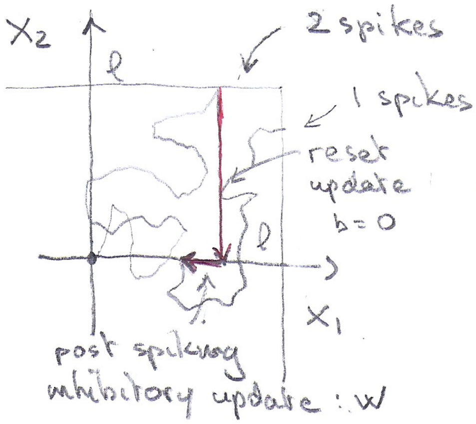
excitation first: spike!

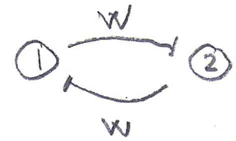
No ordering problem for fully excitatory or fully inhibitory networks.

↳ <u>excitatory</u>: interactions hasten spikes and may cause spiking avalanches.

↳ <u>inhibitory</u>: interactions delay spikes, only one neuron spikes at a time.

Fully inhibitory network





2 inhibitory neurons

ℓ: spiking threshold

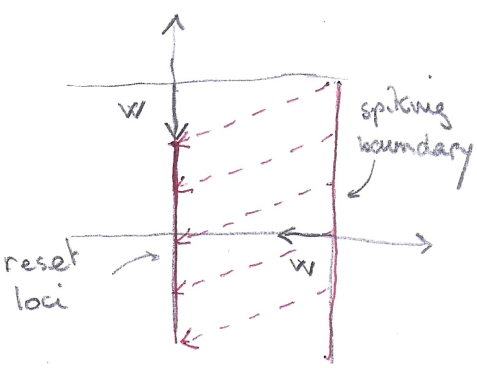
w: synaptic weights . $w < 0$

subthreshold dynamics: 2-dimensional diffusion with infinitesimal generator
$$\mathcal{L} = \underbrace{v_1 \partial_{x_1} + \frac{\sigma_1^2}{2} \partial_{x_1 x_1}}_{\mathcal{L}_1} + \underbrace{v_2 \partial_{x_2} + \frac{\sigma_2^2}{2} \partial_{x_2 x_2}}_{\mathcal{L}_2}$$

neuronal interactions: 1 spikes : reset $X_1 \leftarrow b = 0$

update $X_2 \leftarrow X_2 - w$

network dynamics: diffusion in a space where boundaries are glued to the reset loci.



2-dimensional test function : $f(x, \ell) = f(x-w, 0)$
$f(\ell, x) = f(0, x-w)$

Fokker Planck . $\int_\Omega \mathcal{L}[f]\, p \, dx_1 dx_2 = \int_\Omega \left(\mathcal{L}_1[f] + \mathcal{L}_2[f] \right) p \, dx_1 dx_2$

$$\int_\Omega \mathcal{L}_1[f]\, p \, dx_1 \overset{\text{IBP}}{=} -\int_\Omega \partial_{x_1}[v_1 p] f \, dx_1 - \int_\Omega \partial_{x_1}\left[\frac{\sigma_1^2}{2} p\right] \partial_{x_1} f \, dx_1$$

boundary term

$$= -\int_\Omega \partial_{x_1}[v_1 p] f \, dx_1 \overset{\text{IBP}}{+} \int_\Omega \partial_{x_1 x_1}\left[\frac{\sigma_1^2}{2} p\right] f \, dx_1 - \partial_{x_1}\left[\frac{\sigma_1^2}{2} p\right]\Big|_{x_1 = \ell} f(\ell, x_2)$$

test function

$$\partial_{x_1}\left[\frac{\sigma_1^2}{2}p\right](\ell,x_2)\, f(\ell,x_2) \overset{\downarrow}{=} \underbrace{\partial_{x_1}\left[\frac{\sigma_1^2}{2}p\right](\ell,x_2)}_{-r(x_2)}\, f(0,x_2-w)$$

$-r(x_2)$ ← ~ "Firing rate"

$$= -r_1(x_2) \int_\Omega \delta_0(x_1)\, f(x_1,x_2-w)$$

Free diffusion dual

$$\int_\Omega \mathcal{L}_1[f]\, p\, dx_1\, dx_2 = \int_\Omega \Big[\mathcal{L}_1^+[p]\, f(x_1,x_2) + r_2(x_2)\,\delta_0(x_1)\, f(x_1,x_2-w)\Big]\, dx_1\, dx_2$$

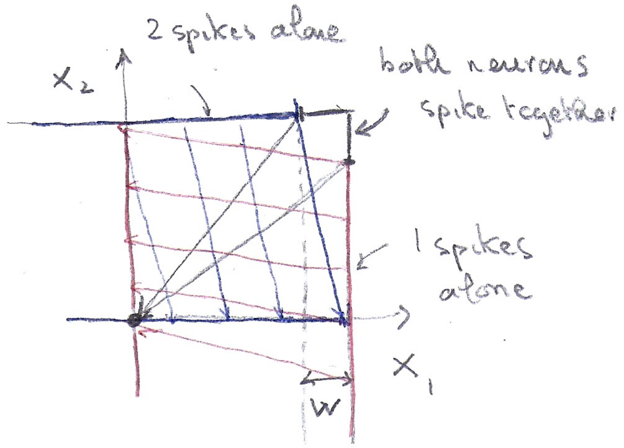
$$= \int_\Omega \underbrace{\Big[\mathcal{L}_1^+[p] + r_1(x_2+w)\,\mathbb{1}_{\{x_2<\ell-w\}}\,\delta_0(x_1)\Big]}_{\mathcal{L}_1^* \,\leftarrow\, \text{dual on } \Omega}\, f(x_1,x_2)\, dx_1\, dx_2$$

$$\boxed{\begin{aligned} \partial_t p &= \mathcal{L}_1^+ p + \mathcal{L}_2^+ p + r_1(x_2+w)\,\mathbb{1}_{\{x_2<\ell-w\}}\,\delta_0(x_1) \\ &\quad + r_2(x_1+w)\,\mathbb{1}_{\{x_1<\ell-w\}}\,\delta_0(x_2) \\ r_2(x) &= \partial_{x_2}\left[\frac{\sigma_2^2}{2}p\right](x,\ell) \end{aligned}}$$

← source term due to post spiking reset

← ~ "Firing rate" of 2

Fully excitatory networks



2 excitatory neurons

⚠ 1 and 2 can spike together and reset to $(0,0)$
↳ there is a Dirac mass at $(0,0)$

network dynamics:

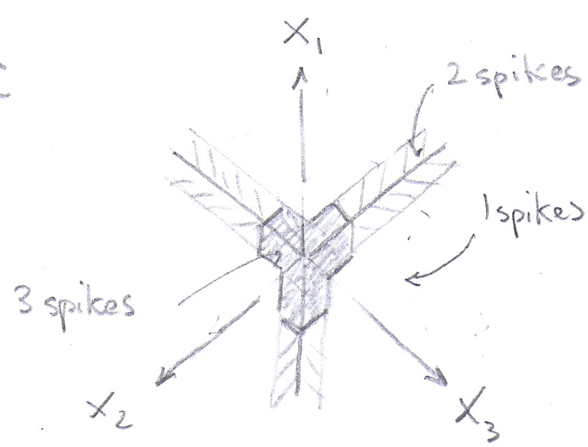
$$\partial_t p = \mathcal{L}_1^+[p] + \mathcal{L}_2^+[p] + r_1(x_2 - w)\,\delta_0(x_1) + r_2(x_1 - w)\,\delta_0(x_2)$$

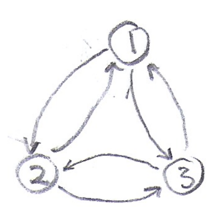
$$+ r_{12}\,\delta_0(x_1)\,\delta_0(x_2)$$ ← 3 source terms

$$r_1(x) = \partial_{x_1}\left[\frac{\sigma_1^2}{2} p\right](\ell, x)$$

$$r_{12} = \int_{\ell - w}^{\ell} \big(r_1(x) + r_2(x)\big)\,dx$$ ← rate of simultaneous spiking

Generalization





→ 3 spiking plane each corresponding to the spiking of a neuron that may trigger others

Mixed excitation inhibition network

Consistant dynamics require to adopt a partition of the N spiking hyperplanes (corresponding each to a neuron) together with consistant update rules.

Example: inhibition is always process first.

Mean-field limit (heuristical approach)

N: number of neurons $\qquad w \leftarrow \frac{W}{N}$ vanishing weight

For N neurons

dependence on non-spiking neurons (update rule)

$$\partial_t p(x_1, \dots, x_N, t) = \sum_i \mathcal{L}_i^+ p + \sum_{k=1}^{N} \sum_{\substack{I \subset \{1,\dots,N\} \\ |I| = k}} \mathcal{G}_I(x_i, i \notin I) \prod_{i \in I} \delta(x_i)$$

spiking pattern of size k
I = set of spiking neuron

reset of spiking neuron

Heuristic: $N \to +\infty \quad w \leftarrow \frac{W}{N}$: no coincidential events pattern for $k \geq 2$ do not happen.

neurons spike independently and only interact via the mean deterministic drive exerted at the population level.

Integration over N-1 variables

Excitatory exchangeable system with constant v and σ

$$\partial_t p_1(x_1;t) = \mathcal{L}_1^+ p_1(x_1;t) + \sum_{i=2}^{N} \frac{\sigma^2}{2}\left[\partial_{x_i} p_{1i}(x_1,\ell;t) - \partial_{x_i} p_{1i}\left(x_1-\frac{w}{N},\ell;t\right)\right]$$

1-D marginal

2-D marginal

$$-\frac{\sigma^2}{2}\partial_{x_1} p_1(x_1;t)\,\delta_0(x_1)$$

exchangeability

$$= \mathcal{L}_1^+ p_1(x_1;t) + \frac{\sigma^2}{2} N\left[\partial_{x_2} p_{12}(x_1,\ell;t) - \partial_{x_2} p_{12}\left(x_1-\frac{w}{N},\ell;t\right)\right]$$

$$-\frac{\sigma^2}{2}\partial_{x_1} p_1(x_1;t)\,\delta_0(x_1)$$

$$\xrightarrow[N\to+\infty]{} \mathcal{L}_1^+ p_1(x_1;t) + w\frac{\sigma^2}{2}\partial_{x_1 x_2} p_{12}(x_1,\ell;t)$$

$$-\frac{\sigma^2}{2}\partial_{x_1} p_1(x_1;t)\,\delta_0(x).$$

$$= -\partial_{x_1}\left\{\left[v - w\frac{\sigma^2}{2}\frac{\partial_{x_2} p_{12}(x_1,\ell;t)}{p_1(x_1;t)}\right] p_1(x_1;t)\right\}$$

$$-\frac{\sigma^2}{2}\partial_{x_1} p_1(x_1;t)\,\delta_0(x)$$

effective drift

Independence ansatz of mean-field limit: $p_{12}(x,y) = p_1(x)\,p_1(y)$

effective drift: $v + \underbrace{w\left(-\frac{\sigma^2}{2}\frac{\cancel{p_1}}{\cancel{p_1}}\partial_x p_1\big|_{x=\ell}\right)}_{\text{rate of firing}}$

Mean-Field equation:

$$\partial_t p = -\partial_x\left\{\left[v - \frac{w\sigma^2}{2}\partial_x p|_{x=\ell}\right] p\right\} + \frac{\sigma^2}{2}\partial_{xx} p$$
$$- \frac{\sigma^2}{2}\partial_x p|_{x=\ell}\, \delta_0(x)$$

Self-consistent non-linear equation (McKean Vlasov)

Existence of solution via propagation of chaos method. (Tanré Delarue 2015)

Stationary solutions: $\frac{\sigma^2}{2}\partial_x p|_{x=\ell} = F = \text{constant}$

$$\partial_t p = -\underbrace{(v + wF)}_{\text{constant drift}}\partial_x p + \frac{\sigma^2}{2}\partial_{xx} p + F\delta_0$$

First-passage problem:

Mean time: $\langle \tau \rangle = \frac{1}{F} = \frac{\ell}{v + wF}$ ← threshold, ← drift $\Rightarrow$ $\boxed{F = \frac{v}{\ell - w}}$

explosion for $w = \ell$ confirmed by simulation

$\boxed{\text{Beyond mean-field ?}}$