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# REPLICA-MEAN-FIELD LIMITS FOR INTENSITY-BASED NEURAL NETWORKS\*

3 FRANÇOIS BACCELLI <sup>†</sup> AND THIBAUD TAILLEFUMIER <sup>‡</sup>

4 **Abstract.** Neural computations emerge from myriads of neuronal interactions occurring in  
5 intricate spiking networks. Due to the inherent complexity of neural models, relating the spiking  
6 activity of a network to its structure requires simplifying assumptions, such as considering models  
7 in the thermodynamic mean-field limit. In the thermodynamic mean-field limit, an infinite number  
8 of neurons interact via vanishingly small interactions, thereby erasing the finite size of interactions.  
9 To better capture the finite-size effects of interactions, we propose to analyze the activity of neural  
10 networks in the replica-mean-field limit. Replica-mean-field models are made of infinitely many  
11 replicas which interact according to the same basic structure as that of the finite network of interest.  
12 Here, we analytically characterize the stationary dynamics of an intensity-based neural network with  
13 spiking reset and heterogeneous excitatory synapses in the replica-mean-field limit. Specifically, we  
14 functionally characterize the stationary dynamics of these limit networks via ordinary differential  
15 equations derived from the Poisson Hypothesis of queuing theory. We then reduce this functional  
16 characterization to a system of self-consistency equations specifying the stationary neuronal firing  
17 rates. Of general applicability, our approach combines rate-conservation principles from point-process  
18 theory and analytical considerations from generating-function methods. We validate our approach  
19 by demonstrating numerically that replica-mean-field models better capture the dynamics of neural  
20 networks with large, sparse connections than their thermodynamic counterparts. Finally, we explain  
21 that improved performance by analyzing the neuronal rate-transfer functions, which saturate due to  
22 finite-size effects in the replica-mean-field limit.

23 **Key words.** neural network, point process, replica model, mean-field theory, Palm calculus,  
24 stochastic intensity.

25 **AMS subject classifications.** 37H10, 37M25, 60K15, 60K25, 90B15, 92B20

26 **1. Introduction.** Intensity-based networks form a natural and flexible class of  
27 models for neural networks, whose study has a long and successful history in compu-  
28 tational neuroscience [48, 21, 60, 47]. In these models, the spiking activity of neural  
29 networks is represented in terms of point processes that are governed by neuronal  
30 “stochastic intensities” [19, 20]. Neuronal stochastic intensities model the instanta-  
31 neous firing rate of a neuron as a function of the spiking inputs received from other  
32 neurons, thereby mediating network interactions and possibly carrying out local com-  
33 putations. Detailed knowledge about intensity-based networks is mostly limited to  
34 simplifying limits such as the thermodynamic limit, i.e., with a very large number of  
35 neurons interacting very weakly [3, 4, 54, 26]. Such an approximation, which neglects  
36 the finite-size of neuronal interactions, precludes explaining and predicting several  
37 aspects of neural computations, including dynamical metastability [2, 57], correlation  
38 regime of activity [32, 37] and modulation of variability [18, 25]. There is a crucial  
39 need for a computational framework allowing for the analysis of structured neural  
40 networks, while taking into account the finiteness of neuronal interactions.

41 Here, we develop a computational framework based on replica-mean-field (RMF)  
42 limits to address this need. RMF limits were introduced in two distinct contexts:

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<sup>†</sup>Department of Mathematics and Department of Electrical and Computer Engineering, University of Texas, Austin, TX ([francois.bacelli@austin.utexas.edu](mailto:francois.bacelli@austin.utexas.edu)).

<sup>‡</sup>Department of Mathematics and Department of Neuroscience, University of Texas, Austin, TX ([ttaillef@austin.utexas.edu](mailto:ttaillef@austin.utexas.edu)).

43 in statistical physics with applications to information-capacity calculations in neuro-  
 44 science [5, 45, 30] and in computer networking to analyze communication networks in  
 45 terms of point processes [61, 50, 10, 8]. We are concerned with the latter approach.  
 46 Instead of considering the finite neural network of interest, this RMF approach con-  
 47 sideres closely related limit networks made of infinitely many replicas with the same  
 48 basic neural structure. The core motivation for considering RMF networks is that,  
 49 under the assumption of independence between replicas, referred to as the ‘‘Poisson  
 50 Hypothesis’’ [50, 51], these networks become analytically tractable. In this work, we  
 51 exploit the Poisson Hypothesis to characterize analytically the long-time limit of a  
 52 class of excitatory, intensity-based networks, called linear Galves-Loch erbach (LGL)  
 53 models. In considering this specific class of networks, our goal is to establish the  
 54 foundation for the RMF computational framework in a simple setting rather than  
 55 aiming at generality.

56 For concreteness, let us introduce the RMF framework for a simple intensity-based  
 57 network, namely the ‘‘counting-neuron’’ model. The counting-neuron model consists  
 58 of a fully-connected network of  $K$  exchangeable neurons with homogeneous synaptic  
 59 weights  $\mu$ . For each neuron  $i$ ,  $1 \leq i \leq K$ , the stochastic intensity  $\lambda_i$  increases by  
 60  $\mu > 0$  upon reception of a spike and reset upon spiking to its base rate  $b$ . Thus, its  
 61 stochastic intensity is  $\lambda_i(t) = b + \mu C_i(t)$ , where  $C_i(t)$  is the number of spikes received  
 62 at time  $t$  since the last reset. Assuming the network state  $\{C_1(t), \dots, C_K(t)\}$  has a  
 63 well-defined stationary distribution, a natural question is: how does the stationary  
 64 firing rate  $\beta = \mathbb{E}[\lambda_i]$  depend on  $\mu$  and  $K$ ? Strikingly, despite its simplicity, direct  
 65 treatment of the model, e.g., via its master Kolmogorov equation, fails to yield an  
 66 exact answer for non-degenerate  $K$  and  $\mu$  [46]. This failure is primarily due to the  
 67 presence of high-order correlations among subsets of neurons. Virtually all available  
 68 results are obtained via a mean-field approximation in the thermodynamic limit, i.e.,  
 69 when letting  $K \rightarrow \infty$  (large networks) and  $\mu \rightarrow 0$  [9] (vanishing interactions). In  
 70 this approximation, high-order correlations disappear at the cost of neglecting the  
 71 finite-size effects of neural constituents [59].

72 In principle, RMF models can incorporate correlations up to a given integer order  
 73  $q$ . In this work, we only consider first-order replica models ( $q = 1$ ), which capture  
 74 first-order statistics. For the counting model and for an integer  $M > 0$ , the  $M$ -replica  
 75 model of first order consists of  $M$  replicas, each comprising  $K$  counting neurons.  
 76 Upon spiking, a neuron  $i$  in replica  $m$ , indexed by  $(i, m)$ , delivers spikes with synaptic  
 77 weight  $\mu$  to the  $K - 1$  neurons  $(j, v_j)$ ,  $j \neq i$ , where the replica destination  $v_j$  is chosen  
 78 uniformly at random. Thus, the probability for two replicas to interact over a finite  
 79 period of time vanishes in the limit  $R \rightarrow \infty$ , which intuitively justifies the Poisson  
 80 Hypothesis. The asymptotic independence between replicas makes a direct analytical  
 81 treatment of the model possible, just as in the traditional thermodynamic mean-field  
 82 (TMF) limit. However, by contrast with the traditional TMF limit, the stationary  
 83 state explicitly depends on the finite-size parameters  $K$  and  $\mu$ . We define the RMF  
 84 model of the counting model as the replica network obtained in the limit of infinitely  
 85 many replicas, namely infinite  $M$  but fixed and finite  $K$ .

86 The Poisson Hypothesis allows one to truncate the correlation terms due to neu-  
 87 ronal interactions in the functional characterization of the stationary state of a single  
 88 replica. For instance, in the counting neuron model, we will show that one can de-  
 89 rive a single ordinary differential equation (ODE) for  $G$ , the probability-generating  
 90 function (PGF) of a neuron count  $C$ :

91 (1.1) 
$$\beta - \mu z G'(z) + (\beta(K-1)(z-1) - b)G(z) = 0.$$

92 The truncation of the correlation terms comes at the cost of introducing the firing  
 93 rate  $\beta$  as an unknown parameter in (1.1). As the ODE (1.1) is otherwise analytically  
 94 tractable, characterizing the RMF stationary state amounts to specifying the unknown  
 95 firing rate  $\beta$ . Then, the challenge of the RMF approach consists in specifying the  
 96 unknown firing rate via purely analytical considerations about a parametric system  
 97 of ODEs. For this model, we will show that in the RMF limit, the stationary firing  
 98 rate  $\beta$  is determined as the unique solution of

$$99 \quad (1.2) \quad \beta = \frac{\mu c^a e^{-c}}{\gamma(a, c)} \quad \text{with} \quad a = \frac{(K-1)\beta + b}{\mu} \quad \text{and} \quad c = \frac{(K-1)\beta}{\mu},$$

100 where  $\gamma$  denotes the lower incomplete Euler Gamma function.

101 Introduced for the counting-neuron model, the analytical strategy presented above  
 102 is at the core of our general RMF approach. In this work, we generalize this strategy  
 103 to first-order replica networks with continuous relaxation of the stochastic intensities  
 104 and with general heterogeneous excitatory synaptic connections. This generalization,  
 105 which is stated in [Theorem 3.8](#), is our main computational result. While establishing  
 106 this result, we develop a general methodology for the stationary analysis of RMF  
 107 models, which we summarize below. We also briefly discuss the relevance of applying  
 108 the RMF limit to neural dynamics.

109

110 **Methodology.** For clarity, we summarize the essential tenets of the methodology  
 111 exposed herein. Even under the simplest assumptions, there are no known analyti-  
 112 cal solutions to the Kolmogorov equations capturing the dynamics of intensity-based  
 113 networks. Instead, one has to resort to analyzing caricatures of the dynamics based  
 114 on some simplifications of its correlation structure. The rate-conservation principle  
 115 (RCP) of Palm calculus offers a systematic way of proposing such simplifications.  
 116 The Palm probability of a stationary point process can be interpreted as the distri-  
 117 bution of this point process conditioned to have a point present at the origin of the  
 118 time axis. The RCP consists in a conservation formula balancing the smooth drift of  
 119 the stationary state variables and their jumps at epochs of the point processes. In  
 120 the RCP formula, jump terms are expectations with respect to Palm probabilities,  
 121 whereas the smooth dynamics leads to expectations with respect to the stationary  
 122 law of the system. Typically, the simplification then consists in replacing these Palm  
 123 expectations by stationary expectations, i.e., in ignoring the Palm bias. The resulting  
 124 simplified functional equations can be solved in some fortunate cases. The key to such  
 125 resolutions is to realize that our simplification of the RCP has a dynamical-system  
 126 interpretation, which can be precisely formulated as a RMF limit. Indeed, the hall-  
 127 mark of RMF limit is to decouple network constituents, thereby cancelling out Palm  
 128 biases. This observation is instrumental in guaranteeing that there is at least one  
 129 probabilistic, physical solution to our simplified functional equations. Such solutions  
 130 are found by imposing analyticity requirements that any probabilistic solution must  
 131 satisfy.

132

133 **Applications.** We do not intend to systematically investigate the applications  
 134 of the RMF approach to neuroscience here, but rather, to highlight two key features  
 135 of the RMF limit. First, we numerically simulate exemplars of recurrent and feedfor-  
 136 ward networks to compare the performance of RMF models and TMF models. We  
 137 show that TMF models outperform TMF models in predicting the neuronal spiking  
 138 rates in LGL networks with strong, sparse synaptic interactions. Second, we perform  
 139 an asymptotic analysis of the neuronal rate-transfer functions, which are determined

140 by the self-consistency equations in both the RMF and TMF limits. This analysis  
 141 shows that the RMF limit fundamentally differs from the classical TMF limit because  
 142 increasing synaptic weights at fixed input rates leads to saturation—an aspect that  
 143 cannot be captured by TMF models which consequently overestimate firing rates. Fur-  
 144 ther applications to neural-network analysis, such as higher-order models, are possible.  
 145 Beyond neuroscience, our methodology is also applicable to generic intensity-based  
 146 stochastic network dynamics. This suggests using the RMF framework to revisit clas-  
 147 sical problems in queuing theory, particle-system theory, communication networks,  
 148 population dynamics, epidemics, as well as completely new problems arising in, e.g.,  
 149 social network dynamics.

150

151 **Structure.** In [Section 2](#), we introduce the point-process modeling framework for  
 152 which we will develop RMF networks. In [Section 3](#), we characterize analytically the  
 153 stationary state of RMF networks. The neuroscience implications and the computa-  
 154 tional relevance of this approach are discussed in [Section 4](#). [Section 5](#) comprises the  
 155 proofs supporting the results presented in [Section 2](#) and in [Section 3](#). Future research  
 156 directions are presented in [Section 6](#), where we explain that similar strategies  
 157 apply for *i*) any correlation orders  $q$ , namely with replica constituents being subsets  
 158 of  $q$  interacting neurons rather than single neurons, and *ii*) for networks with hetero-  
 159 geneous synaptic weights supporting both excitation and inhibition.

160

161 **Related work.** The inspiration for the replica models proposed in this work is  
 162 rooted in the theory of nonlinear Markov processes, which were introduced by McK-  
 163 ean [\[40\]](#). These processes were extensively used to study mean-field limits in queueing  
 164 systems, initially by the Dobrushin school [\[61, 52, 50, 51\]](#), and later by M. Bramson  
 165 [\[11\]](#). This literature has two distinct components: *i*) a probabilistic component prov-  
 166 ing asymptotic independence from the equations satisfied by the non-linear Markov  
 167 process, and *ii*) a computational component deriving closed-form expressions for the  
 168 mean-field limit of the system of interest. These two components jointly led to a  
 169 wealth of new results in queueing theory, concerning both open and closed queueing  
 170 networks, e.g., [\[61\]](#). The aim of this work is to show that, just as in queueing theory,  
 171 studying neural networks in the RMF limit is computationally tractable. Finding  
 172 moment-generating functions by imposing condition of analyticity on some solutions  
 173 is a classical approach in queueing theory [\[56\]](#). The RCP simplification described  
 174 in the methodology summary were used to analyze point-process-based dynamics in  
 175 peer-to-peer networks [\[7\]](#) and in wireless networks [\[53\]](#). However, the link established  
 176 between RMF models and simplified RCP is novel. Our approach also elaborates on  
 177 prior attempts to solve the neural master equations in computational neuroscience.  
 178 Brunel *et al.* introduced mean-field limits for large neural networks with weak inter-  
 179 actions from a computational perspective [\[1, 14, 13\]](#). Touboul *et al.* then adapted the  
 180 ideas of “propagation of chaos” for neural networks in the thermodynamic mean-field  
 181 limit [\[9, 58, 49\]](#). Their results were specialized to spiking models with memory resets  
 182 by Galves and Locherbäch, who also provided perfect algorithms to simulate the sta-  
 183 tionary states of infinite networks [\[29, 22\]](#). Closer to our approach, Buice, Cowan, and  
 184 Chow adapted techniques from statistical physics to analyze the hierarchy of moment  
 185 equations obtained from the master equations [\[16, 17\]](#). These authors were able to  
 186 truncate the hierarchy of moment equations to consider models amenable to finite-size  
 187 analysis via system-size or loop expansion around the mean-field solution [\[12\]](#). These  
 188 authors also showed by field-theoretic arguments that the dynamics obtained by mo-  
 189 ment closure was indeed that of a physical system. Although the master equation of

190 Buice *et al.* does not have a natural small parameter, the moment-closure approach  
 191 was implemented to solve the neural master equations in the thermodynamic limit  
 192 [15]. By contrast, our approach considers a new mean-field regime, that of the RMF  
 193 model for finite-size neural networks, without any scaling of the interactions.

194

195 **2. Point-process framework for finite neural networks.** In this section,  
 196 we introduce the point-process modeling framework for which we will develop RMF  
 197 networks. In [Subsection 2.1](#), we define the intensity-based neural networks that we  
 198 consider throughout this work, i.e., linear Galves-Löcherbach (LGL) networks. In  
 199 [Subsection 2.2](#), we justify that finite LGL networks admit a well-defined stationary  
 200 regime with exponentially integrable neuronal stochastic intensities. In [Subsection 2.3](#),  
 201 we derive functional equations characterizing the stationary joint distribution of the  
 202 neuronal stochastic intensities via the use of the RCP.

203 **2.1. Linear Galves-Löcherbach models.** We consider a finite assembly of  $K$   
 204 neurons whose spiking activity is modeled as the realization of a system of simple point  
 205 processes without common points  $\mathbf{N} = \{N_i\}_{1 \leq i \leq K}$  on  $\mathbb{R}$  defined on an underlying  
 206 measurable space  $(\Omega, \mathcal{F})$ . For all neurons  $1 \leq i \leq K$ , we denote by  $\{T_{i,n}\}_{n \in \mathbb{Z}}$ , the  
 207 sequence of successive spiking times with the convention that almost surely  $T_{i,0} \leq 0 <$   
 208  $T_{i,1}$  and  $T_{i,n} < T_{i,n+1}$ . Each point process  $N_i$  is a family  $\{N_i(B)\}_{B \in \mathcal{B}(\mathbb{R})}$  of random  
 209 variables with values in  $\mathbb{N} \cup \{\infty\}$  indexed by the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  of the real line  
 210  $\mathbb{R}$ . Concretely, the random variable  $N_i(B)$  counts the number of times that neuron  
 211  $i$  spikes within the set  $B$ , i.e.,  $N_i(B) = \sum_{n \in \mathbb{Z}} \mathbb{1}_B(T_{i,n})$ . Setting the processes  $N_i$ ,  
 212  $1 \leq i \leq K$ , to be independent Poisson processes defines the simplest instance of our  
 213 point-process framework as a collection of non-interacting neurons.

214 To model spike-triggered interactions within the network, we consider that the  
 215 rate of occurrences of future spikes depends on the spiking history of the network.  
 216 In other words, we allow the instantaneous firing rate of neuron  $i$  to depend on the  
 217 times at which neuron  $i$  and other neurons  $j \neq i$  have spiked in the past. Formally,  
 218 the network spiking history  $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$  is defined as a non-decreasing family of  $\sigma$ -fields  
 219 such that, for all  $t$ ,

$$220 \quad (2.1) \quad \mathcal{F}_t^{\mathbf{N}} = \{\sigma(N_1(B_1), \dots, N_K(B_K)) \mid B_i \in \mathcal{B}(\mathbb{R}), B_i \subset (-\infty, t]\} \subset \mathcal{F}_t,$$

221 where  $\mathcal{F}_t^{\mathbf{N}}$  is the internal history of the spiking process  $\mathbf{N}$ . The network spiking  
 222 history  $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$  determines the rate of occurrence of future spikes via the notion of  
 223 stochastic intensity. The stochastic intensity of neuron  $i$ , denoted by  $\{\lambda_i(t)\}_{t \in \mathcal{R}}$ , can  
 224 be seen as a function of  $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$  specifying the instantaneous firing rate of neuron  $i$ .  
 225 It is formally defined as the  $\mathcal{F}_t$ -predictable process  $\{\lambda_i(t)\}_{t \in \mathcal{R}}$  satisfying

$$226 \quad \mathbb{E}[N_i(s, t) \mid \mathcal{F}_s] = \mathbb{E}\left[\int_s^t \lambda_i(s) ds \mid \mathcal{F}_s\right],$$

227 for all interval  $(s, t]$  [35]. Stochastic intensities generalize the notion of rate of events,  
 228 or hazard function, to account for generic history dependence beyond that of Poisson  
 229 processes or renewal processes.

230 Specifying the history-dependence of the neuronal stochastic intensities entirely  
 231 defines a network model within the point-process framework. In this work, we con-  
 232 sider models for which the stochastic intensities  $\lambda_1, \dots, \lambda_K$  obey a system of coupled

233 stochastic equations

$$\begin{aligned}
 234 \quad & \lambda_i(t) = \lambda_i(0) + \frac{1}{\tau_i} \int_0^t (b_i - \lambda_i(s)) ds + \\
 235 \quad (2.2) \quad & \sum_{j \neq i} \mu_{ij} \int_0^t N_j(ds) + \int_0^t (r_i - \lambda_i(s)) N_i(ds),
 \end{aligned}$$

236 where the spiking processes  $N_i$  have stochastic intensity  $\lambda_i$ . The above system of  
 237 stochastic equations characterizes the history-dependence of the stochastic intensities.  
 238 The first integral term indicates that in between spiking events,  $\lambda_i$  deterministically  
 239 relaxes toward its base rate  $b_i > 0$  with relaxation time  $\tau_i$ . The second integral terms  
 240 indicates that a spike from neuron  $j \neq i$  causes  $\lambda_i$  to jump by  $\mu_{ij} \geq 0$ , the strength  
 241 of the synapse from  $j$  to  $i$ . Finally, the third integral term indicates that  $\lambda_i$  resets to  
 242  $0 \leq r_i \leq b_i$  upon spiking of neuron  $i$ . Taking  $r_i < b_i$  models the refractory behavior  
 243 of neurons whereby spike generation causes the neuron to enter a transient quiescent  
 244 phase.

245 Thus-defined, our model can be seen as a system of coupled Hawkes processes with  
 246 spike-triggered memory reset and belongs to the Galves-Löcherbach class of models  
 247 [29]. Defining  $T_{i,0}(t)$  to be the last spiking time before time  $t$ , i.e.,  $T_{i,0}(t) = T_{i,0} \circ \theta_t =$   
 248  $\sup\{s \leq t | N_s < N_t\}$ , where  $\theta_t$  is the time-shift operator, the stochastic intensity  $\lambda_i(t)$   
 249 can be written under Galves-Löcherbach form

$$250 \quad (2.3) \quad \lambda_i(t) = \phi_i \left( \sum_j \mu_{ij} \int_{T_{i,0}(t)}^t g_i(t-s) N_j(ds), t - T_{i,0}(t) \right),$$

251 with linear intensity functions  $\phi_i$  and exponentially decaying kernels  $g_i$ :

$$252 \quad (2.4) \quad \phi_i(x, s) = x + b_i + (r_i - b_i)e^{-\frac{s}{\tau_i}} \quad \text{and} \quad g_i(t-s) = e^{-\frac{t-s}{\tau_i}}.$$

253 For this reason, we refer to our model as the linear Galves-Löcherbach model. Galves-  
 254 Löcherbach models have been primarily studied for infinite networks, notably to char-  
 255 acterize the mean-field dynamical limit [22, 23] or to construct perfect simulation  
 256 algorithms [34]. Here, we focus on finite, excitatory assemblies of LGL neurons to  
 257 approximate their dynamics via independent model akin to mean-field models but  
 258 without taking any scaling limit. That being said, we do not consider the proposed  
 259 framework for its biological relevance *per se* as we do not include important aspects  
 260 of neural dynamics such as inhibition. Our goal is rather to develop ideas amenable  
 261 to generalization in a simple setting.

262 **2.2. Stationary Markovian dynamics.** In LGL networks, the stochastic in-  
 263 tensity  $\lambda_i(t)$  determines the instantaneous spiking rate of neuron  $i$  and can be viewed  
 264 as the state of neuron  $i$  at instant  $t$ . When considered collectively, the stochas-  
 265 tic intensities specify the network state  $\boldsymbol{\lambda}(t) = \{\lambda_1(t), \dots, \lambda_K(t)\}$  which follows a  
 266 continuous-time, pure-jump Markovian dynamics with infinitesimal generator

$$267 \quad (2.5) \quad \mathcal{A}[f](\boldsymbol{\lambda}) = \sum_i \frac{b_i - \lambda_i}{\tau_i} \partial_{\lambda_i} f(\boldsymbol{\lambda}) + \sum_i (f(\boldsymbol{\lambda} + \boldsymbol{\mu}_i(\boldsymbol{\lambda})) - f(\boldsymbol{\lambda})) \lambda_i,$$

268 for all  $f$  in  $\mathcal{D}(\mathcal{A})$  the domain of  $\mathcal{A}$ . In the above definition, the first sum collects  
 269 the relaxation terms of the dynamics whereas the second sum corresponds to the

270 interaction jumps triggered by the spiking of neuron  $i$ :

$$271 \quad (2.6) \quad [\boldsymbol{\mu}_i(\boldsymbol{\lambda})]_j = \begin{cases} \mu_{ji} & \text{if } j \neq i \\ r_i - \lambda_i & \text{if } j = i \end{cases} .$$

272 Conditionally to the identity of the spiking neuron, the interaction jumps have fixed  
 273 components set by the synaptic weights and a state-dependent component due to  
 274 spiking reset. The spiking reset to a history-independent state introduces a form of  
 275 degeneracy which substantially hinders the analysis of the network dynamics, espe-  
 276 cially with respect to the regularity of the law of  $\boldsymbol{\lambda}$ . In turn, for lack of a regularity  
 277 characterization, it is unclear how to derive the Kolmogorov forward equation satisfied  
 278 by  $\boldsymbol{\lambda}(t)$  from the Kolmogorov backward equation  $\partial_t u + \mathcal{A}[u] = 0$ .

279 Despite these regularity complications, the stability of the network dynamics  
 280 can be established within the framework of Harris Markov chains [43], whereby the  
 281 continuous-time Markov chain  $\{\boldsymbol{\lambda}(t)\}_{t \in \mathbb{R}}$  proves to be Harris ergodic as long as  $r_i > 0$   
 282 for all neurons  $i$ . As the Markov chain  $\{\boldsymbol{\lambda}(t)\}_{t \in \mathbb{R}}$  is Harris ergodic (see the proof in  
 283 [Subsection 5.1.1](#)), the network dynamics admits a unique invariant measure  $p$  on  $\mathbb{R}^K$   
 284 satisfying

$$285 \quad (2.7) \quad \int_{\mathbb{R}^K} \mathcal{A}[f](\boldsymbol{\lambda}) p(d\boldsymbol{\lambda}) = 0 ,$$

286 for all  $f$  in  $\mathcal{D}(\mathcal{A})$ . Sampling  $\boldsymbol{\lambda}(0)$  according to the stationary measure  $p$  defines the  
 287 stationary version of the Markov chain  $\boldsymbol{\lambda}$ , whose law  $P$  is invariant under time shifts,  
 288 i.e.,  $P \circ \theta_t = P$  for all  $t > 0$ , and whose definition is naturally extended on the whole  
 289 real line  $\mathbb{R}$ . Coupling techniques using Nummelin splittings show that non-stationary  
 290 dynamics converge at least exponentially in total variation toward the stationary limit  
 291 process [33]. The present work is only concerned with the stationary version of the  
 292 network dynamics and, in the following, the notation  $\boldsymbol{\lambda}$  always refers to that stationary  
 293 version. Moreover, processes induced by  $\boldsymbol{\lambda}$ , such as the point processes  $N_i$ , inherit  
 294 the stationary property.

295 We state the technical results justifying the existence of the stationary regime of  
 296 the dynamics in [Section 5](#). A key step is to check a Foster-Lyapunov drift condition in  
 297 [Proposition 5.1](#) for the infinitesimal generator  $\mathcal{A}$  acting on exponential scale functions:  
 298  $V_u(\boldsymbol{\lambda}) = \exp(u \sum_i \lambda_i)$ , where  $u$  is an arbitrary real (see [Subsection 5.1.1](#)). The  
 299 satisfaction of this condition implies that the stationary measure  $p$  is exponentially  
 300 integrable [44]: for all  $u > 0$ , we have

$$301 \quad (2.8) \quad \mathbb{E}[V_u(\boldsymbol{\lambda})] = \int_{\mathbb{R}^K} e^{u \sum_i \lambda_i} p(d\boldsymbol{\lambda}) < \infty .$$

302 Exponential integrability implies the finiteness of the stationary moments of all or-  
 303 ders. Thus, within the context of finite LGL networks, the assumptions of bounded  
 304 intensities function  $\phi_i$  is not required for the existence of stationary moments.

305 *Remark 2.1.* The regularity of the stationary measure of Galves-Locherbach net-  
 306 works has been studied in [38] under assumption of bounded intensity functions  $\phi_i$   
 307 in  $C^\infty(\mathbb{R})$ . In particular, a criterion is given for the stationary measure to admit a  
 308  $C^k(\mathbb{R})$  density with respect to the Lebesgue measure on  $\mathbb{R}$  for finite relaxation times  
 309  $\inf_i \tau_i > 0$ .

310 **2.3. Functional equation for generating functions.** Within the stationary  
 311 framework, it is natural to investigate the relation between low-dimensional features

of the dynamics, such as the moments of the invariant measure, and the structure of the network. In particular, it would be highly desirable to express the individual mean spiking rates, i.e., the average intensities  $\beta_i = \mathbb{E}[N_i((0, 1])]$ , in terms of the model parameters, namely the time constants  $\tau_i$ , the base rates  $b_i$ , the reset values  $r_i$ , and most importantly, the synaptic weights  $\mu_{ij}$ . However, direct analysis of the model via its infinitesimal generator does not provide any tractable characterization of the stationary moments  $\beta_{n_1, \dots, n_K} = \mathbb{E}[\lambda_1^{n_1} \dots \lambda_1^{n_K}]$ . In fact, deriving equations for the moments  $\beta_{n_1, \dots, n_K}$  from the infinitesimal generator would yield a non-closed hierarchy of equations, whereby equations characterizing moments of a given order requires knowledge of moments of higher order [17, 46].

An alternative to such direct approaches consists in looking for equations satisfied by functional transforms of  $p$ , such as the Laplace transform. The reason for considering functional transforms is that at stationarity, one can exploit the RCP [6] to exhibit a functional characterization of these transforms, which can be solved by analytical methods for judiciously chosen functional transforms. In practice, we find that the Laplace transform—or rather the moment-generating function (MGF)—of  $p$  proves the most amenable for the analytical treatment of LGL networks. By exponential integrability of the stationary distribution  $p$  (2.8), the MGF of  $p$

$$(2.9) \quad \mathbf{u} = \{u_1, \dots, u_K\} \mapsto L(\mathbf{u}) = \mathbb{E} \left[ \exp \left( \sum_{i=1}^K u_i \lambda_i \right) \right],$$

is well-defined on all  $\mathbb{R}_+^K$ , and thus characterizes the probability distribution  $p$ . In particular, the moments of  $p$  can be derived from  $L$  as

$$(2.10) \quad m_{n_1, \dots, n_K} = \mathbb{E}[\lambda_1^{n_1} \dots \lambda_1^{n_K}] = \frac{\partial^{\sum_i n_i} L}{\prod_i \partial \lambda_i^{n_i}} \Big|_{\boldsymbol{\lambda}=\mathbf{0}}.$$

The MGF of the stationary distribution  $p$  constitutes the functional transform of choice for the analysis of LGL networks because it admits a simple characterization via RCPs:

**PROPOSITION 2.2.** *The full  $K$ -dimensional MGF  $L$  satisfies the first-order linear PDE*

$$(2.11) \quad \left( \sum_i \frac{u_i b_i}{\tau_i} \right) L - \sum_i \left( 1 + \frac{u_i}{\tau_i} \right) \partial_{u_i} L + \sum_i e^{(u_i r_i + \sum_{j \neq i} u_j \mu_{ji})} \partial_{u_i} L \Big|_{u_i=0} = 0.$$

*Proof.* Given a function  $f$  in the domain  $\mathcal{D}(A)$ , the  $\mathcal{F}_t$ -predictable process defined by

$$(2.12) \quad f(\boldsymbol{\lambda}(t)) - \int_0^t \mathcal{A}[f](\boldsymbol{\lambda}(s)) ds$$

is a martingale. By stationarity of  $\{\boldsymbol{\lambda}(t)\}_{t \in \mathbb{R}}$ , we have  $\mathbb{E}[f(\boldsymbol{\lambda}(t))] = \mathbb{E}[f(\boldsymbol{\lambda}(0))]$  and Dynkin's formula reads

$$(2.13) \quad \mathbb{E} \left[ \int_0^t \mathcal{A}[f](\boldsymbol{\lambda}(s)) ds \right] = \int_0^t \mathbb{E}[\mathcal{A}[f](\boldsymbol{\lambda}(s))] ds = 0.$$



347 Moreover, also by stationarity of  $\{\lambda(t)\}_{t \in \mathbb{R}}$ , the expectation in the integrand is con-  
 348 stant, i.e.,  $\mathbb{E}[\mathcal{A}[f](\lambda(s))] = \mathbb{E}[\mathcal{A}[f](\lambda)]$  with:

$$349 \quad (2.14) \quad \mathbb{E}[\mathcal{A}[f](\lambda)] = \sum_i \mathbb{E} \left[ \frac{b_i - \lambda_i}{\tau_i} \partial_{\lambda_i} f(\lambda) + (f(\lambda + \mu_i(\lambda)) - f(\lambda)) \lambda_i \right] = 0.$$

350 Specializing the above relation to exponential functions  $f(\lambda) = e^{\sum_i u_i \lambda_i}$  yields

$$351 \quad (2.15) \quad \sum_i \mathbb{E} \left[ \frac{b_i - \lambda_i}{\tau_i} u_i e^{\sum_j u_j \lambda_j} + \left( e^{u_i r_i + \sum_{j \neq i} u_j (\lambda_j + \mu_{ji})} - e^{\sum_j u_j \lambda_j} \right) \lambda_i \right] = 0.$$

352 which can be written under the form

$$353 \quad \sum_i \frac{b_i u_i}{\tau_i} \mathbb{E} \left[ e^{\sum_j u_j \lambda_j} \right] - \sum_i \left( 1 + \frac{u_i}{\tau_i} \right) \mathbb{E} \left[ \lambda_i e^{\sum_j u_j \lambda_j} \right]$$

$$354 \quad (2.16) \quad + \sum_i e^{(u_i r_i + \sum_{j \neq i} u_j \mu_{ji})} \mathbb{E} \left[ \lambda_i e^{\sum_{j \neq i} u_j \lambda_j} \right] = 0.$$

355 Equation (2.11) follows from recognizing the expectation terms as values of the MGF  
 356  $L$  and its partial derivatives  $\partial_{\lambda_i} L$ .  $\square$

357 Equation (2.16) is a non-local first-order linear partial differential equation (PDE)  
 358 with boundary terms involving partial derivatives. Conceptually, this equation can  
 359 be viewed as depicting the stationary state of a  $K$ -dimensional transport equation in  
 360 the negative orthant, with linear drift  $(1 + u_1/\tau_1, \dots, 1 + u_K/\tau_K)$ , with linear death  
 361 rate  $\sum_i b_i u_i/\tau_i$ , and with non-local birth rate related to fluxes through the hyperplane  
 362  $\{\lambda_i = 0\}$ ,  $1 \leq i \leq K$ . Despite this conceptual simplicity, the presence of flux-related,  
 363 non-local, birth rate precludes one from solving (2.16) except for the simplest cases,  
 364 i.e., for  $K \leq 2$ . To gain knowledge about the typical state of LGL networks in the sta-  
 365 tionary limit, one has to resort to approximation schemes, such as moment-truncation  
 366 methods, which can yield unphysical solutions without probabilistic interpretations  
 367 and are often analytically intractable [24]. The purpose of the present work is to in-  
 368 troduce a computational framework circumventing the above difficulties by studying  
 369 replica versions of the LGL networks of interest, which admit stationary states that  
 370 are both probabilistically well-posed and analytically tractable.

371 **3. The Replica-mean-field approach.** In this section, we propose to decipher  
 372 the activity of LGL networks via limit networks made of infinitely many replicas with  
 373 the same basic network structure. In [Subsection 3.1](#), we define the RMF limit for  
 374 LGL networks and the associated RMF *ansatz*, a system of ODEs characterizing  
 375 their stationary regime. In [Subsection 3.2](#), we show that in practice, the RMF *ansatz*  
 376 can be derived without explicit reference to the replica framework via a computational  
 377 tool, called Palm calculus. In [Subsection 3.3](#), we reduce the RMF *ansatz* to a set of  
 378 self-consistency equations specifying the stationary neuronal stochastic intensities.

379 **3.1. Replica-mean-field models.** Replica models are first rigorously defined  
 380 for a finite number of replica and admit similar, albeit higher dimensional, functional  
 381 characterization as plain LGL networks. However, in the RMF limit, the Poisson Hy-  
 382 pothesis allows one to truncate correlation terms due to neuronal interaction, yielding  
 383 a set of ODEs characterizing the RMF stationary state.

384 **3.1.1. Finite-replica models.** In order-one replica models, each replica con-  
 385 sists of the same number of neurons as the original LGL networks, denoted by  $K$ ,  
 386 and within each replica, neurons are labelled by a class index  $1 \leq i \leq K$ . For a finite  
 387 model with  $M$  replicas, let  $N_{m,i}$  denote the point process representing the spiking  
 388 activity of the neuron of class  $i$  in replica  $m$ , referred to as neuron  $(m, i)$ . Moreover,  
 389 let  $\{\lambda_{m,i}\}_{1 \leq m \leq M, 1 \leq i \leq K}$ , denote the corresponding stochastic intensity. Instead of in-  
 390 teracting with neurons in the same replica upon spiking, neuron  $(m, i)$  interacts with  
 391 target neurons of classes  $j \neq i$  from independently and uniformly chosen replicas and  
 392 with synaptic weight  $\mu_{ij}$ . Thus, replica models consist in a caricature of the initial  
 393 model where the interactions between neurons are randomized while keeping the finite  
 394 structure of the original network. The finite replica dynamics can be specified via the  
 395 introduction of stochastic processes registering the sequence of neuronal interactions  
 396 across replicas. For all  $1 \leq m \leq M, 1 \leq i \leq K$ , let  $\{v_{m,ij}(t)\}_{t \in \mathbb{R}}$  be stochastic pro-  
 397 cesses such that for every spiking time  $T$ , i.e., for every point of  $N_{m,i}$ , the random  
 398 variables  $\{v_{m,ij}(T)\}_j$  are independent of the past, mutually independent, and uni-  
 399 formly distributed over  $\{1, \dots, M\} \setminus \{m\}$ . Concretely,  $v_{m,ij}$  indicates the index of the  
 400 replica containing the neuron of class  $i$  targeted by neuron  $(m, j)$  upon spiking. Then,  
 401 the stochastic intensities  $\{\lambda_{m,i}\}_{1 \leq m \leq M, 1 \leq i \leq K}$  characterizing the  $M$ -replica dynamics  
 402 of the finite LGL network obey the following system of coupled stochastic equations:

$$403 \quad \lambda_{m,i}(t) = \lambda_{m,i}(0) + \frac{1}{\tau_i} \int_0^t (b_i - \lambda_{m,i}(s)) ds$$

$$404 \quad (3.1) \quad + \sum_{n \neq m} \sum_{j \neq i} \mu_{ij} \int_0^t \mathbf{1}_{\{v_{n,ij}(s)=m\}} N_{n,j}(ds) + \int_0^t (r_i - \lambda_{m,i}(s)) N_{m,i}(ds).$$

405 These equations, which generalize (2.2), entirely define the Markovian dynamics of  
 406 finite replica models for LGL networks. Similarly, the infinitesimal generator (2.5) can  
 407 be generalized to the finite replica setting. To account for randomized interactions, let  
 408 us introduce the  $K$ -dimensional stationary random vectors  $\mathbf{v}_{m,i}$ , defined by  $[\mathbf{v}_{m,i}]_j =$   
 409  $v_{m,ij}(T_{m,i,0})$  if  $j \neq i$  and  $[\mathbf{v}_{m,i}]_i = m$ , taking values in the set of integers

$$410 \quad (3.2) \quad \mathbf{V}_{m,i} = \left\{ \mathbf{v} \in [1 \dots M]^K \mid v_i = m \quad \text{and} \quad v_j \neq m, j \neq i \right\},$$

411 whose cardinality is  $|\mathbf{V}_{m,i}| = (M-1)^{K-1}$ . By definition, the collection of vectors  $\mathbf{v}_{m,i}$ ,  
 412 which indicates the target neurons of neuron  $(m, i)$ , are identically and uniformly  
 413 distributed on the sets  $\mathbf{V}_{m,i}$ . Consequently, the infinitesimal generator for the  $M$ -  
 414 replica Markovian dynamics can be written as

$$415 \quad \mathcal{A}[f_{\mathbf{u}}](\boldsymbol{\lambda}) = \sum_{i=1}^K \sum_{m=1}^M \left( \frac{b_i - \lambda_{m,i}}{\tau_i} \right) \partial_{\lambda_{m,i}} f_{\mathbf{u}}(\boldsymbol{\lambda})$$

$$416 \quad (3.3) \quad + \sum_{i=1}^K \sum_{m=1}^M \frac{1}{|\mathbf{V}_{m,i}|} \sum_{\mathbf{v} \in \mathbf{V}_{m,i}} \left( f(\boldsymbol{\lambda} + \boldsymbol{\mu}_{m,i,\mathbf{v}}(\boldsymbol{\lambda})) - f(\boldsymbol{\lambda}) \right) \lambda_{m,i},$$

417 where the update due to the spiking of neuron  $(m, i)$  is defined by

$$418 \quad (3.4) \quad \left[ \boldsymbol{\mu}_{m,i,\mathbf{v}}(\boldsymbol{\lambda}) \right]_{j,n} = \begin{cases} \mu_{ji} & \text{if } j \neq i, n = v_j, \\ r_i - \lambda_{m,i} & \text{if } j = i, n = v_j, \\ 0 & \text{otherwise.} \end{cases}$$

419 The arguments developed in [Subsection 2.2](#) for the Markovian analysis of plain LGL  
 420 networks naturally extend to finite replica models. In particular,  $M$ -replica networks  
 421 are Harris ergodic and admit a stationary distribution  $p$ . In turn, we can apply the  
 422 RCP of [Subsection 2.3](#) to the stationary  $M$ -replica dynamics to obtain a functional  
 423 characterization for the MGF of  $p$ :

$$\mathbf{u} \mapsto L(\mathbf{u}) = \mathbb{E} \left[ \exp \left( \sum_{m=1}^M \sum_{i=1}^K u_i \lambda_{m,i} \right) \right].$$

425 Specifically, in [Subsection 5.1.2](#), we show the following result on the LGL networks  
 426 defined in [Subsection 2.1](#):

427 **PROPOSITION 3.1.** *For all LGL networks, the  $M$ -replica MGF  $L$  satisfies the first-*  
 428 *order linear PDE*

$$\begin{aligned}
 & \sum_m \sum_i \frac{b_i u_{m,i}}{\tau_i} L(\mathbf{u}) - \sum_m \sum_i \left( 1 + \frac{u_i}{\tau_i} \right) \partial_{\lambda_{m,i}} L(\mathbf{u}) \\
 & + \sum_m \sum_i \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} e^{(u_{m,i} r_i + \sum_{j \neq i} u_{v,j} \mu_{j,i})} L(\mathbf{u}) = 0.
 \end{aligned}$$

431 The above characterization of replica networks is not simpler than that of plain LGL  
 432 networks. However, the expression of the infinitesimal generator [\(3.3\)](#) shows that  
 433 randomized interactions effectively implement an averaging over replicas. In the limit  
 434 of a large number of replicas  $M \rightarrow \infty$ , one expects such an averaging to erase the  
 435 dependence structure of spiking interactions, and to yield independence between repli-  
 436 cas. Numerical simulations support such a mean-field behavior, which is conceptually  
 437 similar to that of the thermodynamic limit, i.e., with  $K \rightarrow \infty$  and vanishing interac-  
 438 tions scaling as  $1/K$ , but retains important features of the finite network structure.  
 439 Intuitively, independence between two replicas emerges from the so-called ‘‘Poisson  
 440 Hypothesis’’ [\[50, 51\]](#): Over a finite period of time, the probability for a particular neu-  
 441 ron to receive a spike from another given neuron scales as  $1/M$ . Thus, as the number  
 442 of replicas increases, interactions between distinct replicas become ever scarcer, lead-  
 443 ing to replica independence. By the same intuition, we expect spiking deliveries to  
 444 distinct replicas to be asymptotically distributed as independent Poisson point pro-  
 445 cesses, which is precisely the Poisson Hypothesis. Proving the validity of the Poisson  
 446 Hypothesis requires to establish the property of propagation of chaos [\[55\]](#) in the limit  
 447 of an infinite number of replicas  $M \rightarrow \infty$ . This is beyond the aims of our analysis.  
 448 Here, we conjecture that the Poisson Hypothesis holds in the limit  $M \rightarrow \infty$ , and  
 449 our goal is to develop the computational framework for the analysis of infinite-replica  
 450 LGL networks, which we refer to as RMF models.

451 **3.1.2. The replica-mean-field *ansatz*.** Under the Poisson Hypothesis, neu-  
 452 rons from distinct replicas of an RMF network spike independently. Here, we show  
 453 that this assumption of independence leads to a simple functional characterization of  
 454 the MGF of a single replica, which we call the RMF *ansatz*. Consider for instance the  
 455 MGF associated to the first replica:

$$\mathbf{u} \mapsto L(\mathbf{u}) = \mathbb{E} \left[ \exp \left( \sum_{i=1}^K u_{i,1} \lambda_{1,i} \right) \right].$$

457 Denoting  $u_{i,1} = u_1$  and  $\lambda_i = \lambda_{1,i}$  for conciseness, the RCP for the  $M$ -replica network  
 458 applied to  $f(\mathbf{u}) = e^{\sum_{i=1}^K u_i \lambda_i}$  (see [Subsection 5.1.2](#)) yields

$$\begin{aligned}
 459 \quad & \sum_{i=1}^K \left( \frac{b_i u_i}{\tau_i} L(\mathbf{u}) - \frac{u_i}{\tau_i} \partial_{u_i} L(\mathbf{u}) \right) + \sum_{i=1}^K (e^{u_i r_i} - 1) \partial_{u_i} L(\mathbf{u}) \Big|_{u_i=0} \\
 460 \quad (3.8) \quad & + \sum_{i=1}^K \sum_{m>1} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(\sum_{j \neq i, v_j=1} u_j \mu_{ji})} - 1 \right) \mathbb{E} \left[ \lambda_{m,i} e^{\sum_{i=1}^K u_i \lambda_i} \right] = 0.
 \end{aligned}$$

461 The above equation would constitute an autonomous ODE for  $L(u)$ , were it not for  
 462 the interactions with replicas  $M > 1$ , as mediated by the last term of (3.8). The  
 463 independence assumption of the Poisson Hypothesis allows us to close (3.8) in the  
 464 limit of an infinite number of replica  $M \rightarrow \infty$ . The first step in this direction is to  
 465 observe that in the limit  $M \rightarrow \infty$ , only certain vectors  $\mathbf{v}$  contribute meaningfully  
 466 to the interaction terms: these are those vectors representing spike deliveries from a  
 467 neuron  $(m, j)$ ,  $m > 1$ , such that only one spike is delivered to the first replica. In  
 468 fact, we elaborate on this observation in [Subsection 5.1.2](#) to show that

$$\begin{aligned}
 469 \quad & \sum_{i=1}^K \sum_{m>1} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(\sum_{j \neq i, v_j=1} u_j \mu_{ji})} - 1 \right) \mathbb{E} \left[ \lambda_{m,i} e^{\sum_{i=1}^K u_i \lambda_i} \right] = \\
 470 \quad (3.9) \quad & \sum_{i=1}^K \sum_{j \neq i} (e^{u_j \mu_{ji}} - 1) \frac{1}{M-1} \sum_{m>1} \mathbb{E} \left[ \lambda_{m,i} e^{\sum_{i=1}^K u_i \lambda_i} \right] + o(1/M).
 \end{aligned}$$

471 By exchangeability of the replicas, all expectation terms in the right-hand side above  
 472 are equal. Moreover, neurons of the same class have identical mean intensities:  $\beta_i =$   
 473  $\mathbb{E}[\lambda_{m,i}]$ . Exploiting the assumption of independence from the Poisson Hypothesis, we  
 474 thus have

$$475 \quad (3.10) \quad \mathbb{E} \left[ \lambda_{m,i} e^{\sum_{i=1}^K u_i \lambda_i} \right] = \mathbb{E}[\lambda_{m,i}] \mathbb{E} \left[ e^{\sum_{i=1}^K u_i \lambda_i} \right] = \beta_i L(\mathbf{u}).$$

476 Using the fact that we also have  $\beta_i = \partial_{u_i} L(\mathbf{u})|_{u_i=0}$ , we can write (3.8) as

$$\begin{aligned}
 477 \quad & \sum_{i=1}^K -\frac{u_i}{\tau_i} \partial_{u_i} L(\mathbf{u}) + \sum_{i=1}^K \left( \frac{u_i b_i}{\tau_i} + \sum_{j \neq i} (e^{u_i \mu_{ij}} - 1) \beta_j \right) L(\mathbf{u}) + \\
 478 \quad (3.11) \quad & (e^{u_i r_i} - 1) \partial_{u_i} L(\mathbf{u}) \Big|_{u_i=0} = 0.
 \end{aligned}$$

479 The above equation is separable. In keeping with the assumption of independence,  
 480 plugging in the product form  $L(\mathbf{u}) = \prod_{i=1}^K L_i(u_i)$  with  $L_i(u_i) = \mathbb{E} \left[ e^{u_i \lambda_i} \right]$  and  $\beta_i =$   
 481  $L'_i(0)$ , yields the final form of the RMF *ansatz*:

482 **DEFINITION 3.2.** *The RMF ansatz for the LGL network of  $K$  neurons specified*  
 483 *by the interaction weights  $\mu_{ij}$ , the relaxation times  $\tau_i$ , the base rates  $b_i$ , and by the*  
 484 *reset values  $r_i$ ,  $1 \leq i \leq K$ , is defined as the system of coupled ODEs:*

$$485 \quad (3.12) \quad -\left(1 + \frac{u}{\tau_i}\right) L'_i(u) + \left( \frac{u b_i}{\tau_i} + \sum_{j \neq i} (e^{u \mu_{ij}} - 1) \beta_j \right) L_i(u) + \beta_i e^{u r_i} = 0.$$

486 Notice that setting  $u \rightarrow 0$  in (3.12) automatically yields  $L'_i(0) = \beta_i$ . Thus, at the  
 487 cost of introducing the mean firing rates  $\beta = \{\beta_1, \dots, \beta_K\}$ , the Poisson Hypothesis  
 488 allows us to write a closed set of ODEs for the one-dimensional MGF  $L_i$ , should  
 489 the RMF *ansatz* be true. However, in the RMF *ansatz*, the mean firing rates  $\beta$  are  
 490 unknown parameters, and the MGF normalization condition,  $L_i(0) = 1$ , does not  
 491 dispel this indetermination. More generally, there is *a priori* no reason for the RMF  
 492 *ansatz* to admit a MGF as a solution. In the following, we show that for the RMF  
 493 *ansatz* to admit a MGF solution,  $\beta$  needs to solve a set of self-consistency equations.

494 We will first account for this result in the special case of the counting-neuron  
 495 model, i.e., for a fully connected network with homogeneous synaptic weights and  
 496 without relaxation:  $\mu_{ij} = \mu$  and  $\tau_i \rightarrow \infty$ . For the counting-neuron model, it is best  
 497 to work with the probability-generating function (PGF) associated to the counting  
 498 vector  $\mathbf{C} = \{C_i, \dots, C_n\}$ :

499 (3.13) 
$$\mathbf{z} \in [0, 1]^K \mapsto G(\mathbf{z}) = \mathbb{E} \left[ \prod_{i=1}^K z_i^{C_i} \right] = L(\ln z_{i_1}, \dots, \ln z_{i_K}),$$

500 rather than with the actual MGF of  $\mathbf{C}$ , still denoted by  $L$ . Specifically, we have:

501 **DEFINITION 3.3.** *The RMF ansatz for the network of  $K$  node counting neuron*  
 502 *network specified by the interaction weight  $\mu$ , and the reset values  $r$ ,  $1 \leq i \leq K$ , is*  
 503 *defined as the ODE:*

504 (3.14) 
$$\beta - \mu z G'(z) + (\beta(K - 1)(z - 1) - r)G(z) = 0.$$

505 Before proceeding to the reduction of the RMF *ansatz* to a set of self-consistency  
 506 equations for  $\beta$ , we show that the RMF *ansatz* can be obtained without any explicit  
 507 reference to replica models. In doing so, our aim is to show that the RMF *ansatz*  
 508 can be established intuitively via independence assumptions, and without in-depth  
 509 probabilistic analysis.

510 **3.2. Functional equations via Palm calculus.** The derivation of the RMF  
 511 *ansatz* relies on a computational tool from the theory of point processes, called Palm  
 512 calculus [39, 41].

513 **3.2.1. Primer on Palm calculus.** Palm calculus treats stationary point pro-  
 514 cesses from the point of view of a typical point, i.e., a typical spike, rather than from  
 515 the point of view of a typical time, i.e., in between spikes. Here, we only introduce  
 516 Palm calculus via the two formulae that play a key role in deriving the RMF *ansatz*  
 517 [6]. With no loss of generality, consider a stationary point process  $N_i$  defined on some  
 518 probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , representing the spiking activity of a neuron. If  $\{\theta_t\}$  is a  
 519 time shift on  $(\Omega, \mathcal{F})$  which preserves  $\mathbb{P}$ , we say that the stationary point process  $N$  is  
 520  $\theta_t$ -compatible in the sense that  $N(B) \circ \theta_t = N(B + t)$  for all  $B$  in  $\mathcal{B}(\mathbb{R})$  and  $t \in \mathbb{R}$ .  
 521 With this notation, the Palm probability of  $N$ , which gives the point of view of a  
 522 “typical” point on  $N$ , is defined on  $(\Omega, \mathcal{F})$  for all event  $A$  in  $\mathcal{F}$  and for all time  $t > 0$   
 523 by

524 (3.15) 
$$\mathbb{P}_N^0(A) = \frac{1}{\beta t} \mathbb{E} \left[ \sum_{n \in \mathbb{Z}} \mathbb{1}_A(\theta_{T_n}) \mathbb{1}_{(0,t]}(T_n) \right] = \frac{1}{\beta t} \mathbb{E} \left[ \int_{(0,t]} (1_A \circ \theta_s) N(ds) \right],$$

525 where  $\beta = \mathbb{E}[N((0, 1])]$ . Informally,  $\mathbb{P}_N^0(A)$  represents the conditional probability  
 526 that a train of spikes falls into  $A$  knowing that a spike happens at  $t = 0$ . Moreover,

527 suppose that  $N$  admits a stochastic intensity  $\lambda_i$ , representing the instantaneous firing  
 528 rate, and set  $A = \{\lambda(0) \in B\}$  for some  $B$  in  $\mathcal{B}(\mathbb{R}_+)$ , then

$$529 \quad (3.16) \quad \mathbb{P}_N^0(A) = \mathbb{P}_N^0[\lambda(0_-) \in B] = \mathbb{P}[\lambda(0_-) \in B \mid N(\{0\}) = 1]$$

530 specifies the stationary law of the stochastic intensity  $\lambda_i$  just before spiking.

531 The notions of Palm probability and stochastic intensity provide the basis for the  
 532 theory of Palm calculus. Let us consider another non-negative stochastic process  $X$   
 533 defined on the same underlying probability space  $(\Omega, \mathcal{F})$  as that of  $N$ . If  $X$  is also  
 534  $\theta_t$ -compatible in the sense that  $X(s) \circ \theta_t = X(s+t)$  for all  $t, s \in \mathbb{R}$ , then the first key  
 535 formula Palm calculus directly follows from the definition (3.15) and reads

$$536 \quad (3.17) \quad \mathbb{E}_N^0[X(0_-)] = \frac{1}{\beta t} \mathbb{E} \left[ \int_0^t X(s) N(ds) \right],$$

537 where  $\mathbb{E}_N^0[\cdot]$  denotes the expectation with respect to  $\mathbb{P}_N^0$ . In the following, the process  
 538  $X$  intervening in the above expression will typically be a function of the stochastic  
 539 intensity of a neuron. The second key formula, which follows from the Papangelou  
 540 theorem, relates Palm probabilities to the underlying probability via the notion of  
 541 stochastic intensity [6]. Specifically, if  $N$  admits a stochastic intensity  $\lambda$  and  $X$  has  
 542 appropriate predictability properties, then for all real valued functions  $f$  we have:

$$543 \quad (3.18) \quad \mathbb{E}[f(X)\lambda_i] = \beta \mathbb{E}_N^0[f(X(0_-))] .$$

544 The formulae (3.17) and (3.18) will be the only results required to establish rate-  
 545 conservation equations via Palm calculus.

546 **3.2.2. Rate-conservation equations.** Because interactions are temporally lo-  
 547 calized at spiking times, Palm calculus is a convenient tool to express rate-conservation  
 548 equations in LGN networks. In fact, Palm calculus allows one to obtain rate-con-  
 549 servation equations intuitively from the stochastic equations describing the evolution  
 550 of the conserved quantity. For our purpose of recovering the RMF *ansatz* from Def-  
 551 inition 3.2, that conserved quantity is  $e^{u\lambda_i}$ , where  $u$  is some fixed real and where  $\lambda_i$   
 552 is the stochastic intensity of neuron  $i$ ,  $0 \leq i \leq K$ . By  $\mathcal{F}_t$ -predictability and sta-  
 553 tionarity of the network dynamics  $\lambda_t$ , for all real  $u$ , the process  $\{e^{u\lambda_i(t)}\}_{t \in \mathbb{R}}$  is also  
 554 a  $\mathcal{F}_t$ -predictable stationary process. Moreover, this process satisfies the stochastic  
 555 equation

$$556 \quad e^{u\lambda_i(t)} = e^{u\lambda_i(0)} + \frac{u}{\tau_i} \int_0^t (b_i - \lambda_i(s)) e^{u\lambda_i(s)} ds$$

$$557 \quad (3.19) \quad + \sum_{j \neq i} (e^{u\mu_{ij}} - 1) \int_0^t e^{u\lambda_i(s)} N_j(ds) + \int_0^t (e^{ur_i} - e^{u\lambda_i(s)}) N_i(ds),$$

558 where the  $N_i$ ,  $0 \leq i \leq K$ , are  $\mathcal{F}_t$ -predictable counting processes with stochastic  
 559 intensity  $\lambda_i$ . In (3.19), the first integral term is due to relaxation toward base rate  
 560  $b_i$ , the second integral term is due to interaction with spiking neurons  $j \neq i$ , and the  
 561 last term is due to post-spiking regeneration of neuron  $i$  at reset value  $r_i$ . Taking  
 562 the expectation of (3.19) with respect to the stationary measure of  $\lambda$  yields the rate-  
 563 conservation equations of  $\{e^{u\lambda_i(t)}\}_{t \in \mathbb{R}}$ :

$$564 \quad \frac{u}{\tau_i} \mathbb{E} \left[ \int_0^t (b_i - \lambda_i(s)) e^{u\lambda_i(s)} ds \right]$$

$$565 \quad (3.20) \quad + \sum_{j \neq i} (e^{u\mu_{ij}} - 1) \mathbb{E} \left[ \int_0^t e^{u\lambda_i(s)} N_j(ds) \right] + \mathbb{E} \left[ \int_0^t (e^{ur_i} - e^{u\lambda_i(s)}) N_i(ds) \right] = 0,$$

566 where we have used that by stationarity, we have  $\mathbb{E}[e^{u\lambda_i(t)}] = \mathbb{E}[e^{u\lambda_i(0)}] = \mathbb{E}[e^{u\lambda_i}]$ .  
 567 Again, by stationarity, the expectation of the relaxation integral term can be expressed  
 568 as

$$569 \quad (3.21) \quad \mathbb{E} \left[ \int_0^t (b_i - \lambda_i(s)) e^{u\lambda_i(s)} ds \right] = t \mathbb{E} [(b_i - \lambda_i) e^{u\lambda_i}] ,$$

570 where  $\beta_i = \mathbb{E}[\lambda_i] = \mathbb{E}[N_i((0, 1])]$  is the mean intensity of  $N_i$ . In turn, introducing  
 571 the Palm distribution  $\mathbb{P}_i^0$  of the process  $\boldsymbol{\lambda}$  with respect to  $N_i$  allows us to write the  
 572 expectations of the remaining interaction and reset integral terms as expectations  
 573 with respect to Palm distributions  $\mathbb{P}_i^0$ ,  $1 \leq i \leq K$ . Specifically, by applying formula  
 574 (3.17), we have

$$575 \quad (3.22) \quad \mathbb{E} \left[ \int_0^t e^{u\lambda_i(s)} N_j(ds) \right] = (\beta_j t) \mathbb{E}_j^0 [e^{u\lambda_i(0^-)}] ,$$

$$576 \quad (3.23) \quad \mathbb{E} \left[ \int_0^t (e^{ur_i} - e^{u\lambda_i(s)}) N_i(ds) \right] = (\beta_i t) \mathbb{E}_i^0 [e^{ur_i} - e^{u\lambda_i(0^-)}] ,$$

577 where  $\mathbb{E}_i^0[\cdot]$  denotes expectation with respect to  $\mathbb{P}_i^0$ . With these observations, the  
 578 rate-conservation equation can be expressed under a local form, i.e., without integral  
 579 terms, but at the cost of taking expectation with respect to distinct probabilities:

$$580 \quad \frac{u}{\tau_i} \mathbb{E} [(b_i - \lambda_i) e^{u\lambda_i}] \\ 581 \quad (3.24) \quad + \sum_{j \neq i} (e^{u\mu_{ij}} - 1) \beta_j \mathbb{E}_j^0 [e^{u\lambda_i(0^-)}] + \beta_i \mathbb{E}_i^0 [e^{ur_i} - e^{u\lambda_i(0^-)}] = 0 .$$

582 The above equation can then be expressed under a local form involving only the  
 583 stationary measure thanks to Papangelou's theorem (3.18), allowing us to write

$$584 \quad (3.25) \quad \beta_j \mathbb{E}_j^0 [e^{u\lambda_i(0^-)}] = \mathbb{E} [\lambda_j e^{u\lambda_i}] \quad \text{and} \quad \beta_i \mathbb{E}_i^0 [e^{u\lambda_i(0^-)}] = \mathbb{E} [\lambda_i e^{u\lambda_i}] .$$

585 Using the above relations in (3.24), the final form of the exact rate-conservation  
 586 equations of  $\{e^{u\lambda_i(t)}\}_{t \in \mathbb{R}}$ ,  $1 \leq i \leq K$ , is

$$587 \quad - \left( 1 + \frac{u}{\tau_i} \right) \mathbb{E} [\lambda_i e^{u\lambda_i}] + \frac{u b_i}{\tau_i} \mathbb{E} [e^{u\lambda_i}] \\ 588 \quad (3.26) \quad + \sum_{j \neq i} (e^{u\mu_{ij}} - 1) \mathbb{E} [\lambda_j e^{u\lambda_i}] + \beta_i e^{ur_i} = 0 ,$$

589 where we have dropped time dependence for stationary random variables.

590 **3.2.3. Moment truncation.** Applying the RCP under the Poisson Hypoth-  
 591 esis effectively truncates correlation terms due to interactions in the exact rate-  
 592 conservation equation of replica models. Although not apparent in the Markovian  
 593 treatment of Subsection 3.1.2, such a truncation become straightforward when work-  
 594 ing on the rate-conservation equation (3.26) obtained via Palm calculus. Indeed,  
 595 (3.26) can be interpreted as a differential equation for the one-dimensional MGF of  
 596  $\boldsymbol{\lambda}$  defined by  $L_i(u) = \mathbb{E}[e^{u\lambda_i}]$  for all  $i$ . However, (3.26) for  $L_i$  involves the second-  
 597 order statistics of  $\boldsymbol{\lambda}$  via the terms  $\mathbb{E}[\lambda_j e^{u\lambda_i}]$ , which is not captured by  $L_i$  but by the  
 598 two-dimensional MGFs of  $\boldsymbol{\lambda}$ . Not surprisingly, making the Poisson Hypothesis allows

599 one to close (3.26), as it implies that the stochastic intensities of distinct neurons are  
 600 independent variables:

$$601 \quad (3.27) \quad \mathbb{E} [\lambda_j e^{u\lambda_i}] = \beta_j \mathbb{E} [e^{u\lambda_i}] \quad \text{for } j \neq i.$$

602 Thus, under the Poisson Hypothesis, (3.26) becomes an equation about the random  
 603 variable  $\lambda_i$  alone:

$$604 \quad - \left( 1 + \frac{u}{\tau_i} \right) \mathbb{E} [\lambda_i e^{u\lambda_i}] \\
 605 \quad (3.28) \quad + \left( \frac{ub_i}{\tau_i} + \sum_{j \neq i} (e^{u\mu_{ij}} - 1) \beta_j \right) \mathbb{E} [e^{u\lambda_i}] + \beta_i e^{ur_i} = 0.$$

606 The above equation is precisely that intervening in the mean-field-replica *ansatz*  
 607 in Definition 3.2. As announced, it has been obtained by truncation of the rate-  
 608 conservation equations via Palm calculus and without any explicit reference the RMF  
 609 network. Considering (3.28) as a heuristic simplification of (3.26) leads to a natural  
 610 question: why should the heuristic simplification based on (3.27) lead to some equation  
 611 having a probabilistic interpretation? The RMF framework provides the answer to  
 612 this question: the RMF network is a stochastic dynamical system whose steady-  
 613 state MGF should satisfy (3.28). In other words, the existence of a steady state for  
 614 the RMF network, which is conjectured here, justifies the existence of at least one  
 615 probabilistic solution to (3.28). As stated previously, proving rigorously the existence  
 616 of that steady state consists in establishing the property of propagation of chaos [55]  
 617 in RMF networks, which is beyond the aims of our analysis.

618 **3.3. Analytical solutions for replica-mean-field models.** The rate-con-  
 619 servation equations appearing in the RMF *ansatz* are first-order ODEs. Hence, char-  
 620 acterizing the stationary state of RMF networks amounts to specifying the unknown  
 621 mean intensities featuring in these differential equations. Intuitively, the mean inten-  
 622 sities must solve a set of self-consistency equations: for each neuron,  $\beta_i$  is the output  
 623 firing rate of a neuron subjected to input firing rates  $\beta_j$  delivered via synaptic weight  
 624  $\mu_{ij}$ . The goal of this section is twofold: first, we derive such self-consistency equa-  
 625 tions via simple analyticity requirements of the solutions of the differential equations.  
 626 Second, we numerically validate the properties of the RMF framework by comparison  
 627 with the original LGL network or with the classical thermodynamic limit.

628 **3.3.1. The counting model case.** The analytical strategy that we will follow  
 629 for general LGL models is first exemplified on the simplest network, i.e., the counting  
 630 model with  $K$  fully connected neurons with homogeneous synaptic weights  $\mu$  and  
 631 with uniform base rate  $b$ . By neuronal exchangeability, the RMF *ansatz* for the  
 632 counting model (see Definition 3.3) takes the form of a single equation for the PGF  
 633 of  $C$ , the number of spikes received by a neuron since the last reset. Then, for any  
 634  $\beta$ , that equation admits a unique solution  $G$  satisfying the normalization condition  
 635 that  $G(1) = 1$ , thereby defining a family of candidate PGFs  $\{G_\beta\}_\beta$ , parameterized  
 636 by the unknown  $\beta$ . As explained above, the RMF *ansatz* should have at least one  
 637 solution  $G_\beta$  which is a PGF. It turns out that, for the counting model, requiring the  
 638 analyticity of the solutions in zero is enough to determine a unique PGF solution to  
 639 the RMF *ansatz*. Specifically, we show in the following that, given the normalization  
 640 condition  $G(1) = 1$ , there is a unique continuous solution to the RMF *ansatz* and that  
 641 the normalization condition for that solution yields the self-consistency equation for



642  $\beta$ . Moreover, we are able to show that this equation uniquely specifies  $\beta$  and that the  
 643 corresponding function  $G_\beta$  is indeed a PGF by explicitly exhibiting the associated  
 644 stationary probability distribution. These results are summarized in the following  
 645 theorem:

646 **THEOREM 3.4.** *For the counting model, there is a unique integer-valued random*  
 647 *variable  $C$  whose PGF is solution to the RMF ansatz [Definition 3.3](#). Moreover, (i)*  
 648 *the mean intensity  $\beta = b + \mu\mathbb{E}[C]$  is the unique solution to:*

$$649 \quad (3.29) \quad \beta = \frac{\mu c^a e^{-c}}{\gamma(a, c)} \quad \text{with} \quad a = \frac{(K-1)\beta + b}{\mu} \quad \text{and} \quad c = \frac{(K-1)\beta}{\mu},$$

650 where  $\gamma$  denotes the lower incomplete Gamma function, and (ii) the stationary dis-  
 651 tribution of  $C$  is given by

$$652 \quad p(n) = \begin{cases} \frac{\beta}{\mu a} = \frac{\beta}{(K-1)\beta + b}, & \text{if } n = 0, \\ \frac{c^a e^{-c}}{\gamma(a, b)} \frac{\Gamma(a+n+1)c^n}{\Gamma(a)\Gamma(n+1)}, & \text{if } n > 0. \end{cases}$$

653 (3.30)

654 *Proof.* The unique solution to the first-order differential equation [\(3.14\)](#) that sat-  
 655 isfies the normalization condition  $G(1) = 1$  is

$$656 \quad (3.31) \quad G(z) = \frac{e^{c(z-1)}}{z^a} \left( 1 + \frac{\beta e^c}{\mu c^a} (\Gamma(a, c) - \Gamma(a, cz)) \right),$$

657 where  $\Gamma$  denotes the upper incomplete Gamma function, i.e.,  $\Gamma(x, y) = \int_y^\infty t^{x-1} e^{-t} dt$ ,  
 658 and where we have used the auxiliary parameters  $a$  and  $c$  defined in [\(3.29\)](#). Solutions  
 659  $G$  are analytic on  $\mathbb{R}$  except possibly in zero, where  $G$  generically has an infinite  
 660 discontinuity. Indeed, noting that  $a > 0$ , we have the following asymptotic behavior  
 661 when  $z \rightarrow 0^+$ :

$$662 \quad (3.32) \quad G(z) = z^{-a} \left( e^{-c} + \frac{\beta(\Gamma(a, c) - \Gamma(a))}{\mu b^a} \right) + \frac{\beta}{\mu a} + O(z).$$

663 As probability-generating functions must be analytic in zero, we require the term be-  
 664 tween parentheses to be zero in the above expression, which is equivalent to requiring  
 665 that  $\beta$  solves the leftmost equation of [\(3.29\)](#). Observing that  $(K-1)\beta = c/\mu$  and  
 666  $a = c + b/\mu$ , [\(3.29\)](#) can be rewritten as an equation on  $c$ :

$$667 \quad (3.33) \quad c^{1-(c+\frac{b}{\mu})} e^c \gamma\left(c + \frac{b}{\mu}, c\right) = K - 1$$

668 Then, applying [Lemma 3.5](#) (see below) with  $x = b/\mu$  and  $y = K - 1$  shows that  
 669 Equation [\(3.29\)](#) admits a unique solution for  $b > 0$ ,  $\mu > 0$  and  $K > 0$ . The result for  
 670  $\mu = 0$ , i.e., for independent neurons, is clear:  $\lambda = b$ . For  $\beta$  solving [\(3.29\)](#), the solution  
 671 to [\(3.14\)](#) can be written

$$672 \quad (3.34) \quad G(z) = \frac{e^{c(z-1)}}{z^a} \frac{\gamma(a, zc)}{\gamma(a, c)},$$

673 and repeated differentiations shows that  $G$  is the PGF associated to the distribution  
 674 defined over the integers by

$$675 \quad (3.35) \quad p(n) = \frac{G^{(n)}(0)}{n!} = \frac{c^a e^{-c}}{\gamma(a, b)} \frac{c^n/n!}{a(a+1)\dots(a+n)} = \frac{c^a e^{-c}}{\gamma(a, b)} \frac{\Gamma(a+n+1)c^n}{\Gamma(a)\Gamma(n+1)},$$

676 and for which we have

$$677 \quad (3.36) \quad p(0) = \frac{\beta}{\mu a} = \frac{\beta}{(K-1)\beta + b} \leq 1. \quad \square$$

678 The proof of [Theorem 3.4](#) utilizes the following lemma:

679 **LEMMA 3.5.** *For all  $x, y \geq 0$ , there is a unique positive real  $c$  such that*

$$680 \quad (3.37) \quad c^{1-(x+c)} e^c \gamma(x+c, c) = y,$$

681 where  $\gamma$  denotes the lower incomplete Gamma function.

682 *Proof.* The power series representation of the incomplete Gamma function yields

$$683 \quad (3.38) \quad f(c) = c^{1-(x+c)} e^c \gamma(x+c, c) = \sum_{n=0}^{\infty} \frac{c^{n+1}}{(x+c)(x+c+1)\dots(x+c+n)},$$

684 where the series converges uniformly in  $c$  on all compacts in  $\mathbb{R}_+$ . Denoting the con-  
 685 tinuous summand functions by

$$686 \quad (3.39) \quad f_n(c) = \frac{c^{n+1}}{(x+c)(x+c+1)\dots(x+c+n)},$$

687 we observe that  $f_n$  is differentiable on  $\mathbb{R}_+^*$  with

$$688 \quad (3.40) \quad f'_n(c) = \frac{c^n \left( n+1 - \sum_{m=0}^n \frac{c}{x+c+m} \right)}{(x+c)(x+c+1)\dots(x+c+n)} > 0.$$

689 Thus, by uniform convergence,  $f$  is a strictly increasing continuous function. To  
 690 prove the lemma, we need to show that  $f$  is onto  $\mathbb{R}_+$ , i.e. that  $\lim_{c \rightarrow \infty} f(c) = \infty$   
 691 since  $f(0) = 0$ . This limit directly follows from the positivity of  $f_n$  on  $\mathbb{R}_+$  and from  
 692 the fact that  $\lim_{c \rightarrow \infty} f_n(c) = 1$  for all  $n \geq 0$ .  $\square$

693 *Remark 3.6.* The generating function  $G$  obtained by solving for  $\mu = 0$  and  $\beta = b$

$$694 \quad (3.41) \quad G(z) = \frac{b}{b + (K-1)(1-z)},$$

695 is the PGF of a geometric distribution with parameter  $(1 + (K-1)/b)^{-1}$ , which is  
 696 precisely the law of independent Poissonian arrivals during an exponential waiting  
 697 time, i.e., the law of the spike count of a neuron during the inter-spike period of  
 698 another. In particular, the mean count value is  $G'(z) = (K-1)/b$ , as expected.

699 *Remark 3.7.* While neglecting coupling between neurons, the stationary distribu-  
 700 tion  $p$  incorporates self-excitation via interaction-dependent mean intensities and also  
 701 captures the effect of spiking reset. For instance, keeping  $a - c = b/\mu$  and letting  
 702  $a \rightarrow \infty$ , as in the limit of large  $K$ , we have

$$703 \quad (3.42) \quad \frac{b^a e^{-b}}{\Gamma(a) - \Gamma(a, b)} = \sqrt{\frac{2a}{\pi}} + O(1),$$

704 which implies an asymptotic scaling law with the network size  $K$  for finite synaptic  
 705 weight  $\mu$ :

$$706 \quad (3.43) \quad \beta \sim \sqrt{\frac{2K\mu\beta}{\pi}} \quad \text{i.e.} \quad \beta \sim \frac{2K\mu}{\pi}.$$

707 **3.3.2. The relaxing model case.** The arguments proving [Theorem 3.4](#) for the  
 708 counting-neuron model essentially generalize to the RMF *ansatz* for heterogeneous  
 709 LGL networks with relaxation (see [Definition 3.2](#)), albeit with some caveats. Indeed,  
 710 we show that the RMF *ansatz* reduces to a set of self-consistency equations by writing  
 711 down that normalization conditions for the set of continuous solutions to the *ansatz*.  
 712 We also show that continuous solutions are necessarily completely monotone, which  
 713 implies by Bernstein's theorem [\[27\]](#), that such solutions are indeed MGF for some  
 714 probability distributions. Moreover, utilizing monotonicity arguments, we show that  
 715 [Theorem 3.4](#) implies the existence of a solution  $\beta$  to the obtained set of self-consistency  
 716 equations. The main caveat is that we do not have any direct argument establishing  
 717 the uniqueness of solutions, although we conjecture that uniqueness holds for hetero-  
 718 geneous LGL networks with relaxation. These results are summarized in the following  
 719 theorem, which is proved in [Subsection 5.2](#):

720 **THEOREM 3.8.** *For all LGL relaxing models, there is a set of independent real ran-*  
 721 *dom variables  $\{\Lambda_i\}_{1 \leq i \leq K}$  whose MGFs  $\{L_i\}_{1 \leq i \leq K}$  are solutions to the RMF ansatz*  
 722 *specified in [Definition 3.2](#) with*

$$723 \quad (3.44) \quad L_i(u) = \beta_i \int_{-\infty}^u \exp \left( \left[ h_i(x) + \sum_{j \neq i} \beta_j h_{ij}(x) \right]_v^u + l_i(v) \right) dv,$$

724 where the functions  $g_i$ ,  $h_i$ , and  $h_{ij}$  are defined by

$$725 \quad (3.45) \quad l_i(x) = \tau_i r_i \left( e^{\frac{x}{\tau_i}} - 1 \right), \quad h_i(x) = b_i \left( \tau_i \left( e^{\frac{x}{\tau_i}} - 1 \right) - x \right),$$

$$726 \quad (3.46) \quad h_{ij}(x) = \tau_i e^{-\tau_i \mu_{ij}} \left( \text{Ei} \left( \tau_i \mu_{ij} e^{\frac{x}{\tau_i}} \right) - \text{Ei} \left( \tau_i \mu_{ij} \right) \right) - x,$$

727 and where  $\text{Ei}$  denotes the exponential integral function. In particular, the mean in-  
 728 tensities  $\mathbb{E}[\Lambda_i] = \beta_i$ ,  $1 \leq i \leq k$ , solve the system of equations

$$729 \quad (3.47) \quad \frac{1}{\beta_i} = \int_{-\infty}^0 \exp \left( -h_i(v) - \sum_{j \neq i} \beta_j h_{ij}(v) + l_i(v) \right) dv.$$

730

731 **Remark 3.9.** The RMF *ansatz* for neurons with excitatory random interaction  
 732 weights and random reset values takes the same form as in [Definition 3.2](#):

$$733 \quad (3.48) \quad - \left( 1 + \frac{u}{\tau_i} \right) L_i'(u) + f_i(u) L_i(u) + g_i(u) = 0.$$

734 but with the functions

$$735 \quad (3.49) \quad f_i(u) = -\frac{ub_i}{\tau_i} + \sum_{j \neq i} \left( 1 - \int_0^\infty e^{u\mu} dq_{ij}(\mu) \right) \beta_j,$$

$$736 \quad (3.50) \quad g_i(u) = \beta_i \int_0^\infty e^{ur} dq_i(r),$$

737 where  $q_{ij}$  is the probability measure of synaptic weight  $\mu_{ij}$  and  $q_i$  is the probability  
 738 measure of the reset  $r_i$ . The above functions  $f_i$  and  $g_i$  still satisfy the key properties  
 739 (see Proposition 5.5) establishing Theorem 3.8, which therefore extends straightfor-  
 740 wardly to the case of excitatory random interactions and random reset values.

741 *Remark 3.10.* The system of equations (3.47) can be interpreted probabilistically  
 742 by considering an isolated relaxing-neuron  $i$  subjected to independent Poissonian de-  
 743 liveries from other neurons with rate  $\beta_j$ . Actually, one can check that the spiking  
 744 activity of such a neuron defines a renewal process with a renewal distribution that  
 745 satisfies

$$746 \quad (3.51) \quad \mathbb{P}[S_i > t] = \exp\left(-h_i(-t) - \sum_{j \neq i} \beta_j h_{ij}(-t) + l_i(-t)\right).$$

747 Then, the set of self-consistency equations (3.47) follows from writing:

$$748 \quad (3.52) \quad \frac{1}{\beta_i} = \mathbb{E}[S_i] = \int_0^\infty \mathbb{P}[S_i > t] dt.$$

749 *Remark 3.11.* In the absence of relaxation, the inhomogeneous model becomes  
 750 the “counting-synapse model”, for which the stochastic intensities can be written as  
 751  $\lambda_i(t) = b_i + \sum_{j \neq i} \mu_{ij} C_{ij}(t)$  via the introduction of the processes

$$752 \quad (3.53) \quad C_{ij}(t) = \int_{T_{i,0}(t)}^t N_j(ds), \quad j \neq i,$$

753 which count the number of spikes that a neuron  $i$  receives from another neuron  $j$  since  
 754 the last time neuron  $i$  spiked. Taking the limit  $\tau_i \rightarrow \infty$  in (3.45) and (3.46) yields to  
 755 the functions  $g_i$ ,  $h_i$ , and  $h_{ij}$  for the counting-synapses model

$$756 \quad (3.54) \quad l_i(x) = r_i x, \quad h_i(x) = 0, \quad \text{and} \quad h_{ij}(x) = \frac{e^{\mu_{ij}x} - 1}{\mu_{ij}} - x,$$

757 where the reset value  $r_i$  coincides with the base rate ( $r_i = b_i$ ).

758 **4. Neuroscience applications.** The aim of this section is to illustrate the con-  
 759 crete applications of the RMF approach through a few examples in neuroscience.  
 760 Since the main tool currently used for this class of problems is the TMF limit, we  
 761 first compare the TMF and the RMF models on a few basic network topologies and  
 762 show how the latter outperforms the former. A fundamental difference between the  
 763 TMF and the RMF is then discussed through the analysis of the so-called transfer  
 764 functions of the two models.

765 **4.1. Numerical comparison with the thermodynamic limit.** At the core  
 766 of the RMF approach is the assumption that the dynamics of finite-size LGL networks  
 767 is well-approximated by neurons experiencing independent Poissonian bombardments  
 768 from other neurons. As already mentioned, another possible simplifying assumption  
 769 is that of the classical TMF limit. In the TMF model, one substitutes an individual  
 770 neuron  $i$  with a population of  $M$  exchangeable neurons with connections weights  
 771  $\mu_{ji}/M$ , and takes the limit of infinite population size  $M \rightarrow \infty$ . Propagation of chaos  
 772 holds in the TMF limit [22]. Thus, a neuron within population  $i$  experiences neuronal  
 773 interactions via the time-dependent deterministic drive

$$774 \quad (4.1) \quad \alpha_i(t) = \sum_{j \neq i} \mu_{ij} \left( \int_0^\infty \lambda p_j(t, \lambda) d\lambda \right),$$

775 where  $p_j(t, \lambda)$  is the probability distribution of the stochastic intensity  $\lambda$  of a neuron  
 776 within population  $j$  at time  $t$ . As a result, all neurons become independent in the  
 777 TMF limit, and each time-dependent probability distribution  $p_i$  satisfy a forward  
 778 Kolmogorov equation that can be written

$$779 \quad \partial_t p_i(t, \lambda_i) = -\partial_{\lambda_i} \left[ \left( \frac{b_i - \lambda_i}{\tau_i} + \alpha_i(t) \right) p_i(t, \lambda_i) \right] -$$

$$780 \quad (4.2) \quad \lambda_i p_i(t, \lambda_i) + \left( \int_0^\infty \lambda p_i(t, \lambda) d\lambda \right) \delta_{r_i}(\lambda_i).$$

781 In the above right-hand side, the first term represents the deterministic drift incorpo-  
 782 rating relaxation and interaction contributions, the second term is a death term due  
 783 to neuronal spiking with rate  $\lambda_i$ , and the last term represents a birth term localized  
 784 at reset value  $r_i$  with population-level rate  $\int_0^\infty \lambda p_i(t, \lambda) d\lambda$ . Introducing the variables  
 785  $s_i = b_i + \tau_i \sum_{j \neq i} \mu_{ij} \beta_j$ , the stationary distribution  $p_i$  is thus solution to the equation

$$786 \quad (4.3) \quad \partial_{\lambda_i} \left[ \left( \frac{s_i - \lambda_i}{\tau_i} \right) p(\lambda_i) \right] + \lambda_i p_i(\lambda_i) = \beta_i \delta_{r_i}(\lambda_i).$$

787 The stationary distribution solving the above equation can be expressed in closed  
 788 form as

$$789 \quad (4.4) \quad p_i(\lambda) = \frac{e^{\tau_i(\lambda - r_i)}}{|s_i - \lambda|} \Big|_{s_i - r_i}^{\tau_i s_i} \beta_i \tau_i \mathbb{1}_{[r_i, s_i]}(\lambda),$$

790 where  $\mathbb{1}_{[r_i, s_i]}$  is the indicator function of the interval  $[r_i, s_i]$ . In turn, the MGF asso-  
 791 ciated to the stationary distribution  $p_i$  can be evaluated as

$$792 \quad (4.5) \quad L_i(u) = \int e^{u\lambda} p_i(\lambda) d\lambda = \frac{\beta_i \tau_i e^{s_i u + (s_i - r_i) \tau_i \gamma(\tau_i s_i, (s_i - r_i)(\tau_i + u))}}{((s_i - r_i)(\tau_i + u))^{\tau_i s_i}},$$

793 from which we deduce the set of TMF self-consistency equations from the normaliza-  
 794 tion conditions  $L_i(0) = 1$ :

$$795 \quad (4.6) \quad \frac{1}{\beta_i} = \frac{\tau_i e^{(s_i - r_i) \tau_i \gamma(\tau_i s_i, (s_i - r_i) \tau_i)}}{((s_i - r_i) \tau_i)^{\tau_i s_i}}.$$

796 Observe that the above self-consistency equations closely mirror the form of the set  
 797 of equations (3.47) obtained from the RMF *ansatz*.

798 To explore the formal correspondence between the RMF and TMF frameworks,  
 799 let us consider RMF models in the thermodynamic limit. In considering such a  
 800 limit, our goal is to evidence how TMF models and first-order RMF models differ.  
 801 Applying the RCP to networks where we substitute each neuron with a population of  
 802  $M$  exchangeable neurons yields the following RMF *ansatz*:

$$803 \quad (4.7) \quad - \left( 1 + \frac{u}{\tau_i} \right) L'_i(u) + \left( \frac{ub_i}{\tau_i} + \sum_{j \neq i} M \left( e^{\frac{u\mu_{ij}}{M}} - 1 \right) \beta_j \right) L_i(u) + \beta_i e^{ur_i} = 0.$$

804 Taking the thermodynamic limit, one has  $\lim_{M \rightarrow \infty} M (\exp(u\mu_{ij}/M) - 1) = u\mu_{ij}$  and  
 805 we obtain the new *ansatz*

$$806 \quad (4.8) \quad - \left( 1 + \frac{u}{\tau_i} \right) L'_i(u) + \frac{us_i}{\tau_i} L_i(u) + \beta_i e^{ur_i} = 0.$$

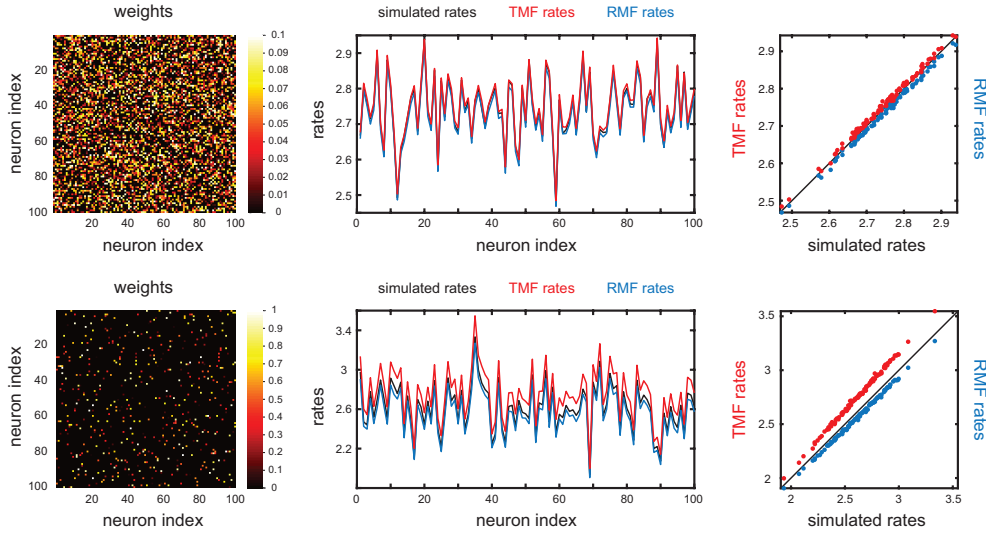


FIG. 1. **Recurrent network.** RMF models better capture the stationary firing rate of finite LGL networks than TMF models for unstructured random networks with sparse, large, synaptic weights. **Top row:** LGL network of 100 counting-synapse neurons, each receiving spikes from randomly sampled 50 neurons, via identically uniformly distributed synaptic weights. **Bottom row:** LGL network of 100 neurons, each receiving spikes from 5 randomly sampled neurons, via identically uniformly distributed synaptic weights. **Left:** Synaptic structure. **Middle:** Numerical stationary rates obtained from discrete-event simulations ( $10^7$  spiking events) and from iterated schemes for the RMF model and TMF model (20 iterations). **Right:** Scatter plots comparing the faithfulness of the TMF model and that of the RMF model.

807 We refer to the above system of equations as the TMF *ansatz*. As expected, one  
 808 can check that the MGFs defined by relation (4.5) are solutions to the TMF *ansatz*.  
 809 Moreover, the difference between TMF models and first-order MGF effectively appears  
 810 to be due to the terms mediating interactions: these terms are exponential in the first-  
 811 order RMF limit, whereas they linearize in the TMF limit.

812 Moreover, we present numerical results emphasizing when the first-order RMF  
 813 approach approximates finite LGL networks more faithfully than TMF networks. We  
 814 consider two types of counting-synapse models (see Remark 3.11): unstructured re-  
 815 current networks in Figure 1 and multilayered feedforward networks in Figure 2. For  
 816 each network structure, we numerically evaluate the empirical stationary firing rates  
 817 of finite LGL networks via discrete-event simulations using the Gillespie algorithm  
 818 [31]. Then, we compare these empirical rates with the RMF rates and the TMF  
 819 rates, which are obtained by numerically solving the self-consistency equations (3.47)  
 820 and (4.6), respectively. These solutions are computed via the—empirically uncondi-  
 821 tionally converging—iteration scheme deduced from the self-consistency equations.  
 822 As expected from our discussion of the TMF limit, Figure 1a and Figure 2a show  
 823 that RMF models closely mirror TMF models for LGL networks with weak interac-  
 824 tions. e.g., with  $\mu_{ij}/b_i \ll 1$ . Moreover, TMF models, as well as RMF models, are both  
 825 faithful approximations of the corresponding finite LGL networks, which exhibit weak  
 826 correlations by construction. Because of the role played by the interaction-mediating  
 827 terms in the TMF and RMF *ansätze*, we expect that RMF models become distinct  
 828 from TMF models for network structure involving large synaptic weights, e.g., with

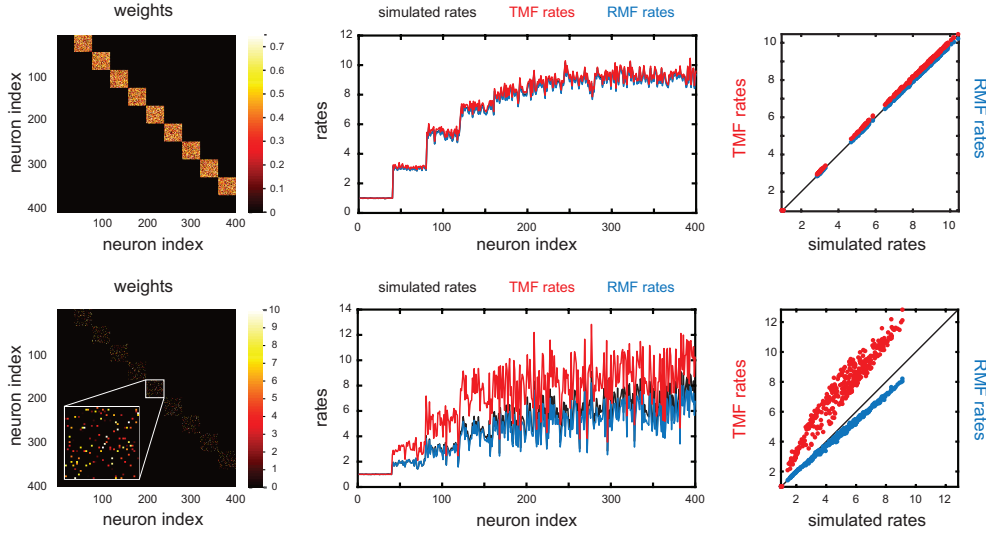


FIG. 2. **Feedforward network.** *RMF models better capture the stationary firing rate of finite LGN networks than TMF models for multilayered feedforward network with sparse, large, synaptic weights.* **Top row:** LGL network of 10 layers of 40 counting-synapse neurons, each receiving spikes from 40 randomly sampled neurons from the previous layer (except the driving layer), via identically uniformly distributed synaptic weights. **Bottom row:** LGL network of 10 layers of 40 neurons, each receiving spikes from 3 randomly sampled neurons from the previous layer (except the driving layer), via identically uniformly distributed synaptic weights. **Left:** Synaptic structure. **Middle:** Numerical stationary rates obtained from discrete-event simulations ( $10^7$  spiking events) and from iterated schemes for the RMF model and TMF model (20 iterations). **Right:** Scatter plots comparing the faithfulness of the TMF model and that of the RMF model.

829  $\mu_{ij}/b_i > 1$ . However, we expect RMF model to be faithful only when the Poisson Hy-  
 830 pothesis is a good modeling assumption, i.e., when spike trains are nearly Poissonian  
 831 and independent across neurons. For large synaptic weights, such a behavior is the  
 832 hallmark of sparsely connected networks. Figure 1b and Figure 2b confirm that RMF  
 833 networks better predict the firing rates of LGL networks with large, sparse, synaptic  
 834 connections. Further numerical simulations reveal that RMF models comparatively  
 835 better capture feedforward networks than recurrent networks (see Table 1). This is  
 836 due to the presence of cycles in the network structure, which promotes correlation and  
 837 gradually invalidates the Poisson Hypothesis [42]. Accounting for networks with large,  
 838 sparse, synaptic connections but strong recurrent structure, e.g., nearest-neighbor lat-  
 839 tice graph, requires to consider higher-order RMF models (see Section 6).

TABLE 1

Comparison of the relative errors of the mean firing rates in the TMF limit and in the RMF limit for different network structures. The RMF limit comparatively better captures the mean firing rates for LGL networks with large, sparse, synaptic connections.

Network model	TMF error	RMF error
Complete unstructured	< 1%	< 1%
Sparse unstructured	5%	2%
Complete feedforward	2%	1%
Sparse feedforward	44%	7%

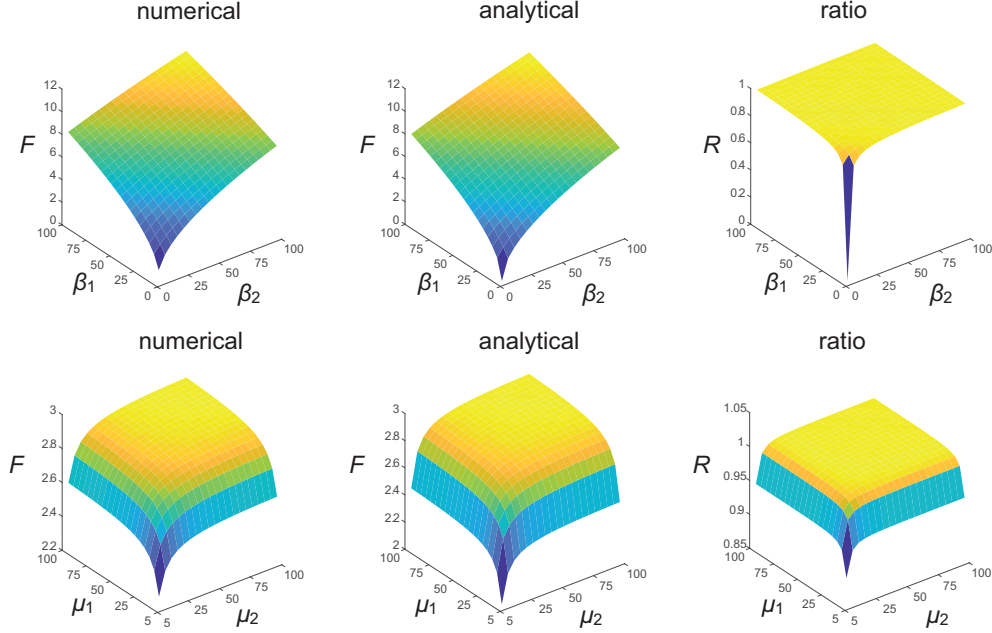


FIG. 3. **Transfer function.** Asymptotic regime of the transfer function  $F$  for a neuron with reset value  $r = 1$ , base level  $b = 1$ , time constant  $\tau = 1$ , and receiving spikes from two other neurons. **Top row.** Numerical and analytical approximation of the transfer function  $F$  for large input rates with synaptic weights  $\mu_1 = \mu_2 = 1$ . A purely excitatory LGL network is always stable because its transfer function grows sublinearly as a function of its input rates. **Bottom row.** Numerical and analytical approximation of the transfer function  $F$  for large synaptic weights with input rates  $\beta_1 = \beta_2 = 1$ . The transfer function saturates for large synaptic weights showing the non-symmetric role of synaptic weights and input rates.

840 **4.2. Asymptotic transfer functions.** A key quantity determining the behav-  
 841 ior of neural networks is the neuronal rate-transfer function, which relates the output  
 842 stationary rate of a neuron to its stationary input rates and its synaptic weights. For  
 843 instance, neurons modeled via Hawkes processes—which neglect reset mechanisms—  
 844 have rate-transfer functions that depend linearly on the rates of interaction  $\mu_{ij}\beta_j$ .  
 845 Such a linear dependence of rate-transfer functions implies that Hawkes neural net-  
 846 works are prone to explosion in the absence of inhibition, and thus fail to admit a  
 847 stationary regime. By contrast, LGL networks are unconditionally stable, indicating  
 848 that the LGL rate-transfer function must grow sublinearly with input rates. Within  
 849 the RMF framework, the rate-transfer function of a neuron  $i$ , denoted  $F_i$ , is given by  
 850 the self-consistency equations (3.47) and can be expressed as

$$851 \quad F_i(\boldsymbol{\beta}, \boldsymbol{\mu}) = \left( \int_0^{\tau_i} \exp \left( -\tilde{h}_i(v) - \sum_{j \neq i} \beta_j \tilde{h}_{ij}(v) \right) \tilde{l}_i(v) dv \right)^{-1},$$

852 where the auxiliary functions  $\tilde{h}_{ij}$ ,  $\tilde{h}_i$  and  $\tilde{l}_i$  are defined as:

$$853 \quad (4.9) \quad \tilde{h}_{ij}(v) = \int_0^v \frac{1 - e^{-\mu_{ij}u}}{1 - u/\tau_i} du, \quad \tilde{h}_i(v) = \frac{b_i}{\tau_i} \int_0^v \frac{u}{1 - u/\tau_i} du, \quad \tilde{l}_i(v) = \frac{e^{-r_i v}}{1 - v/\tau_i}$$



854 (see Equation 5.53). In Figure 3, we numerically compute the rate-transfer function  
 855 of a neuron subjected to two spiking streams with varying input rates and varying  
 856 synaptic weights. Considering the asymptotic behavior of  $F_i$  via the Laplace method  
 857 in the limit of large input rates  $\beta_j$  exhibits the sublinearity of  $F_i$ . Specifically, observ-  
 858 ing that the function  $\tilde{h}_{ij}$  admits its minimum over  $(0, \tau_i)$  in 0, the Laplace method  
 859 implies that in the limit of large input rates, i.e., for all  $\beta_j \rightarrow \infty$ , we have

$$860 \quad (4.10) \quad F_i(\boldsymbol{\beta}, \boldsymbol{\mu})^{-1} \sim e^{-\tilde{h}_i(0) - \sum_{j \neq i} \beta_j \tilde{h}_{ij}(0)} \tilde{l}_i(0) \int_0^\infty e^{-\sum_{j \neq i} \beta_j \tilde{h}_{ij}''(0) v^2 / 2} dv.$$

861 The evaluation of the Gaussian integral with  $\tilde{h}_{ij}''(0) = \mu_{ij}$  yields the asymptotic  
 862 behavior

$$863 \quad (4.11) \quad F_i(\boldsymbol{\beta}, \boldsymbol{\mu}) = \left( \frac{2}{\pi} \sum_{j \neq i} \mu_{ji} \beta_j \right)^{1/2} + o\left(\sqrt{\beta_1}, \dots, \sqrt{\beta_K}\right),$$

864 showing that LGL rate-transfer functions scale with the square-root of the input rates,  
 865 which is consistent with the reset-enforced unconditional stability of LGL networks.  
 866 Such a sublinear scaling is the same as that of the counting-neuron model because  
 867 relaxation becomes irrelevant at high firing rate, i.e., when interspike intervals become  
 868 shorter than the relaxation time constant  $\tau_i$  (see Figure 3).

869 Finally, by contrast with Hawkes model—and with LGL neurons in the TMF  
 870 limit—, the rate-transfer function  $F_i$  exhibits a distinct nonlinear dependence on the  
 871 synaptic weights at fixed input rates. Indeed, we have

$$872 \quad (4.12) \quad \tilde{h}_{ij}(v) = - \sum_{j \neq i} \frac{1}{\mu_{ij}} + O(1/\mu_1^2, \dots, 1/\mu_K^2),$$

873 Then, taking the limit  $\mu_{ij} \rightarrow \infty$  in (4.9) shows that the rate-transfer function  $F_i$   
 874 asymptotically saturates to the upper bound

$$875 \quad (4.13) \quad \bar{\beta}_i = \frac{e^{-a} a^b}{\tau_i \gamma(b, a)} \quad \text{with} \quad a = \tau_i(b_i - r_i) \quad \text{and} \quad b = \tau_i \left( b_i + \sum_{j \neq i} \beta_j \right).$$

876 This upper bound simplifies to  $\bar{\beta}_i = b_i + \sum_{j \neq i} \beta_j$  when the reset level and the base  
 877 level identical:  $b_i = r_i$ . Finally, accounting for first-order corrections shows that for  
 878 large synaptic weights, we have the scaling

$$879 \quad (4.14) \quad F_i(\boldsymbol{\beta}, \boldsymbol{\mu})^{-1} \sim \int_0^{\tau_i} e^{\frac{av}{\tau_i} + \sum_{j \neq i} \frac{\beta_j}{\mu_{ij}}} (1 - v/\tau_i)^{b-1} dv = e^{\sum_{j \neq i} \frac{\beta_j}{\mu_{ij}}} / \bar{\beta}_i,$$

880 so that the rate-transfer function  $F_i$  has the following asymptotic behavior

$$881 \quad (4.15) \quad F_i(\boldsymbol{\beta}, \boldsymbol{\mu}) = \bar{\beta}_i \left( 1 - \sum_{j \neq i} \frac{\beta_j}{\mu_{ij}} \right) + o(1/\mu_1, \dots, 1/\mu_K).$$

882 This saturating behavior is a distinct feature of RMF limit models (see Figure 3).  
 883 Informally, in the limit of infinite weights, each spiking input triggers a spiking output  
 884 leading to an effective quasi-linear transfer function. By contrast, in the TMF limit,  
 885 increasing synaptic weight  $\mu_{ij}$  is equivalent to increasing input rate  $\beta_j$ , so that the  
 886 rate-transfer function diverges in the limit of large synaptic weights. This failure to  
 887 capture saturation in the TMF limit explains why RMF models outperforms TMF  
 888 models for sparse networks with large synaptic weights.

889 **5. Proofs.** This section contains the proofs of the key results of our RMF com-  
 890 putational framework. [Subsection 5.1](#) contains the Markovian analysis justifying the  
 891 Harris ergodicity of LGL networks and their finite replica versions ([Subsection 5.1.1](#))  
 892 and the derivation of the RMF *ansatz* ([Subsection 5.1.2](#)). [Subsection 5.2](#) proves  
 893 [Theorem 3.8](#) solving the RMF *ansatz* for the relaxing-neuron model with synaptic  
 894 heterogeneity.

895 **5.1. Markovian analysis.** Establishing Harris ergodicity, as well as deriving  
 896 the RMF *ansatz*, essentially rely on the Markovian analysis of the infinitesimal gen-  
 897 erators of LGL networks and their finite replica versions.

898 **5.1.1. Harris ergodicity.** To prove Harris ergodicity, it is enough to exhibit a  
 899 regeneration set that is positive recurrent for  $\{\mathbf{\Lambda}_n\}_{n \in \mathbb{Z}}$ , the embedded Markov chain  
 900 of  $\{\boldsymbol{\lambda}(t)\}_{t \in \mathbb{R}}$ , defined as  $\{\mathbf{\Lambda}_n\}_{n \in \mathbb{Z}} = \{\boldsymbol{\lambda}_{T_n}\}_{n \in \mathbb{Z}}$ , where  $T_n$  denotes the ordered sequence  
 901 of jumps such that almost surely  $T_0 \leq 0 < T_1$  and  $T_n < T_{n+1}$ . In [\[49\]](#), Robert and  
 902 Touboul exploit the Poissonian embedding of intensity-based network models [\[36\]](#) to  
 903 show that all compact sets  $R_{\lambda_0} = [0, \lambda_0]^K$  with

$$904 \quad (5.1) \quad \lambda_0 > \max_i \left( \sum_j \mu_{ji} + b_i \right),$$

905 are regeneration sets for  $\{\mathbf{\Lambda}_n\}_{n \in \mathbb{Z}}$ . Briefly, regeneration happens when each neuron  
 906 spikes consecutively and “spontaneously”, i.e., in the base-rate component of the Pois-  
 907 sonian embedding, which is well defined as long as  $\min_i \inf_t \lambda_i(t) = \min_i r_i > 0$ . Given  
 908 an initial state  $\mathbf{\Lambda}_0$  in  $R_{\lambda_0}$ , such a sequence of  $K$  transitions yields a state  $\mathbf{\Lambda}_K$  that  
 909 is independent of  $\mathbf{\Lambda}_0$ , while happening with finite, albeit small, probability. Know-  
 910 ing the regenerative property of compact sets  $R_{\lambda_0}$ , the Harris ergodicity of  $\{\boldsymbol{\lambda}(t)\}_{t \in \mathbb{R}}$   
 911 follows from the existence of positive recurrent compact sets under the assumption  
 912 of a non-explosive behavior. The non-explosive nature of the dynamics, as well as  
 913 the positive recurrence of compact sets  $R_{\lambda_0}$  for large enough  $\lambda_0$ , are established by  
 914 verifying the following Foster-Lyapunov drift condition for exponential scale functions  
 915  $V_u(\boldsymbol{\lambda}) = \exp(u \sum_i \lambda_i)$ :

916 **PROPOSITION 5.1.** *For  $u > 0$  and  $c > 0$ , there are real numbers  $d > 0$  and  $l > 0$   
 917 such that for all  $\lambda_0 > l$  and for all  $\boldsymbol{\lambda}$  in  $\mathbb{R}_+^K$ , we have*

$$918 \quad (5.2) \quad \mathcal{A}[V_u](\boldsymbol{\lambda}) \leq -cV(\boldsymbol{\lambda}) + d\mathbb{1}_{R_{\lambda_0}}(\boldsymbol{\lambda}).$$

919 *Proof.* On  $\mathbb{R}_+^K$ , the infinitesimal increment of the scale function  $V_u$  satisfies

$$920 \quad (5.3) \quad \mathcal{A}[V_u](\boldsymbol{\lambda}) = \sum_i \frac{b_i - \lambda_i}{\tau_i} u V_u(\boldsymbol{\lambda}) + \sum_i \left( e^{u(\sum_{j \neq i} \mu_{ji} + r_i - \lambda_i)} - 1 \right) \lambda_i V_u(\boldsymbol{\lambda}),$$

$$921 \quad (5.4) \quad \leq \left( u \sum_i \frac{b_i}{\tau_i} + \frac{1}{u} \sum_i e^{u(\sum_{j \neq i} \mu_{ji} + r_i) - 1} - \sum_i \lambda_i \right) V_u(\boldsymbol{\lambda}).$$

922 where we used the facts that  $\lambda_i \geq 0$  and that  $\max_{\lambda} e^{-\lambda_i u} \lambda = 1/ue$  for  $u > 0$ . Given  
 923  $c > 0$ , the compact set

$$924 \quad (5.5) \quad R_c = \left\{ \boldsymbol{\lambda} \in \mathbb{R}_+^K \mid \sum_i \lambda_i \leq u \sum_i \frac{b_i}{\tau_i} + \frac{1}{u} \sum_i e^{u(\sum_{j \neq i} \mu_{ji} + r_i) - 1} + c \right\}$$

925 is such that  $\mathcal{A}[V_u] \leq -cV_u$  outside  $R_c$ . Thus, choosing

$$926 \quad (5.6) \quad l = u \sum_i \frac{b_i}{\tau_i} + \frac{1}{u} \sum_i e^{u(\sum_{j \neq i} \mu_{ji} + r_i) - 1} + c,$$

927 implies that, for  $\lambda_0 > l$ ,  $\mathcal{A}[V_u] \leq -cV_u$  outside of  $R = [0, \lambda_0]^K \supset R_c$ . Moreover, using  
928 the boundedness of  $V_u$  on compact sets to choose

$$929 \quad (5.7) \quad d = \lambda_0 \sup_{\lambda \in R} V_u(\lambda) < \infty,$$

930 we finally check that  $\mathcal{A}[V_u] \leq -cV_u + d\mathbb{1}_R$  on  $\mathbb{R}_+^K$ .  $\square$

931 In [44], Meyn and Tweedie show that the Foster-Lyapunov drift condition of  
932 Proposition 5.1 has two immediate implications: *i*) As the functions  $V_u$  are positive  
933 and norm-like, i.e.  $\lim_{\lambda \rightarrow \infty} V_u(\lambda) = \infty$  for  $u > 0$ , Proposition 5.1 directly implies  
934 that the Markovian dynamics is non-explosive. *ii*) As the dynamics is non-explosive  
935 and noting that  $V_u \geq 1$  on  $\mathbb{R}_+^K$ , a set  $R_{\lambda_0}$  satisfying Proposition 5.1 is positive recur-  
936 rent, and for large enough  $\lambda_0$ ,  $R_{\lambda_0}$  is a regeneration set as well, implying the Harris  
937 ergodicity of the Markov chain  $\{\lambda(t)\}_{t \in \mathbb{R}}$ .

938 **5.1.2. Functional equations for replica models.** Following the exact same  
939 steps as for the proof of Proposition 2.2, Dynkin's formula applied at stationarity  
940 allows one to functionally characterize the stationary state of the  $M$ -replica model as  
941 stated in Proposition 3.1.

942 *Proof of Proposition 3.1.* Given a subset of replica indices  $S \subset \{1, \dots, M\}$ , let  
943 us express the infinitesimal generator  $\mathcal{A}$  defined by expression (3.3) for the  $M$ -replica  
944 model when acting on the exponential function

$$945 \quad (5.8) \quad f_{\mathbf{u}}(\lambda) = \exp \left( \sum_{i=1}^K \sum_{m \in S} u_{m,i} \lambda_{m,i} \right).$$

946 We obtain the relation

$$947 \quad \mathcal{A}[f_{\mathbf{u}}](\lambda) = \sum_{i=1}^K \sum_{m \in S} \left( \frac{b_i - \lambda_{m,i}}{\tau_i} \right) u_{m,i} f_{\mathbf{u}}(\lambda)$$

$$948 \quad + \sum_{i=1}^K \sum_{m \in S} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{u_{m,i}(r_i - \lambda_{m,i}) + \sum_{j \neq i, v_j \in S} u_{v_j, j} \mu_{ji}} - 1 \right) f_{\mathbf{u}}(\lambda) \lambda_{m,i}$$

$$949 \quad (5.9) \quad + \sum_{i=1}^K \sum_{m \notin S} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{\sum_{j \neq i, v_j \in S} u_{v_j, j} \mu_{ji}} - 1 \right) f_{\mathbf{u}}(\lambda) \lambda_{m,i}.$$

950 By Dynkin's formula, we have  $\mathbb{E}[\mathcal{A}[f_{\mathbf{u}}](\lambda)] = 0$  for stationary  $M$ -replica dynamics,  
951 which implies that

$$952 \quad 0 = \sum_{i=1}^K \sum_{m \in S} \left( \frac{b_i u_{m,i}}{\tau_i} L(\mathbf{u}) - \frac{u_{m,i}}{\tau_i} \partial_{u_{m,i}} L(\mathbf{u}) \right)$$

$$953 \quad + \sum_{i=1}^K \sum_{m \in S} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(u_{m,i} r_i + \sum_{j \neq i, v_j \in S} u_{v_j, j} \mu_{ji})} - 1 \right) \partial_{u_{m,i}} L(\mathbf{u}) \Big|_{u_{m,i}=0}$$

$$954 \quad (5.10) \quad + \sum_{i=1}^K \sum_{m \notin S} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(\sum_{j \neq i, v_j \in S} u_{v_j, j} \mu_{ji})} - 1 \right) \mathbb{E}[\lambda_{m,i} f_{\mathbf{u}}(\lambda)],$$

955 where we use the notation

$$956 \quad (5.11) \quad L(\mathbf{u}) = \mathbb{E} \left[ \exp \left( \sum_{i=1}^K \sum_{m \in S} u_{m,i} \lambda_{m,i} \right) \right].$$

957 Specifying the above relation for  $S = \{1, \dots, M\}$  yields the PDE of [Proposition 3.1](#). $\square$

958 In the remaining of this section, we justify relation (3.9) used for heuristically  
959 deriving the RMF *ansatz* of [Definition 3.2](#). Considering only the first replica  $S = \{1\}$ ,  
960 and denoting  $u_{1,j} = u_j$  for simplicity, relation (5.10) becomes

$$961 \quad (5.12) \quad 0 = \sum_{i=1}^K \left( \frac{b_i u_i}{\tau_i} L(\mathbf{u}) - \frac{u_i}{\tau_i} \partial_{u_i} L(\mathbf{u}) \right) \\ 962 \quad + \sum_{i=1}^K \frac{1}{|V_{i,1}|} \sum_{\mathbf{v} \in V_{i,1}} \left( e^{(\sum_{j \neq i, v_j=1} u_j \mu_{ji} + u_i r_i)} - 1 \right) \partial_{u_i} L(\mathbf{u})|_{u_i=0} \\ 963 \quad + \sum_{i=1}^K \sum_{m>1} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(\sum_{j \neq i, v_j=1} u_j \mu_{ji})} - 1 \right) \mathbb{E} [\lambda_{m,i} f_{\mathbf{u}}(\boldsymbol{\lambda})].$$

964 As  $\mathbf{v} \in V_{i,1}$  implies  $v_j \neq 1$  for all  $j \neq i$ , the exponent in the second term of the  
965 right-hand side is actually independent of  $\mathbf{v}$  so that we have:

$$966 \quad (5.13) \quad 0 = \sum_{i=1}^K \left( \frac{b_i u_i}{\tau_i} L(\mathbf{u}) - \frac{\lambda_i}{\tau_i} \partial_{u_i} L(\mathbf{u}) \right) \\ 967 \quad + \sum_{i=1}^K (e^{u_i r_i} - 1) \partial_{u_i} L(\mathbf{u})|_{u_i=0} \\ 968 \quad + \sum_{i=1}^K \sum_{m>1} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(\sum_{j \neq i, v_j=1} u_j \mu_{ji})} - 1 \right) \mathbb{E} [\lambda_{m,i} f_{\mathbf{u}}(\boldsymbol{\lambda})].$$

969 By exchangeability of replicas, the value of the expectation term above is independent  
970 of  $m > 1$ . Then, conditionally to neuron  $i$  spiking, let us estimate the sum:

$$971 \quad (5.14) \quad \sum_{m>1} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{(\sum_{j \neq i, v_j=1} u_j \mu_{ji})} - 1 \right) = \frac{(M-1)S_{i,2}}{|V_{i,2}|},$$

972 where  $S_{i,2}$  collects the terms corresponding to interactions with the second replica:

$$973 \quad (5.15) \quad S_{i,2} = \sum_{\mathbf{v} \in V_{i,2}} \left( e^{\sum_{j \neq i, v_j=1} u_j \mu_{ji}} - 1 \right).$$

974 To further estimate  $S_{i,2}$ , observe that the set  $V_{i,2}$  can be partitioned according to how  
975 many of its components are equal to one. Specifically, we have the partition

$$976 \quad (5.16) \quad V_{i,2} = V_{i,2}^{(0)} \cup \dots \cup V_{i,2}^{(K-1)},$$

977 where the non-overlapping sets  $V_{i,2}^{(k)}$ ,  $0 \leq k \leq K-1$ , are defined as

$$978 \quad (5.17) \quad V_{i,2}^{(k)} = \{\mathbf{v} \in V_{i,2} \mid |\{v_j = 1\}| = k\} \quad \text{with} \quad |V_{i,2}^{(k)}| = \binom{K}{k} (M-2)^{K-1-k}.$$

979 Noticing that  $\exp\left(\sum_{j \neq i, v_j=1} u_j \mu_{ji}\right) - 1 = 0$  on  $V_{i,2}^{(0)}$ , we have

$$\begin{aligned}
 980 \quad S_{i,2} &= \sum_{k=1}^{K-1} \sum_{\mathbf{v} \in V_{i,2}^{(k)}} \left( e^{\sum_{j \neq i, v_j=1} u_j \mu_{ji}} - 1 \right) \\
 981 \quad (5.18) &= (M-2)^{K-2} \sum_{j \neq i} (e^{u_j \mu_{ji}} - 1) + (M-2)^{K-3} \sum_{j, k \neq i} (e^{u_j \mu_{ji} + u_k \mu_{ki}} - 1) + \dots
 \end{aligned}$$

982 Remembering that  $|V_{m,i}| = (M-1)^{K-1}$ , we conclude that when  $M \rightarrow \infty$ , we have

$$983 \quad (5.19) \quad \sum_{m>1} \frac{1}{|V_{m,i}|} \sum_{\mathbf{v} \in V_{m,i}} \left( e^{\left(\sum_{j \neq i, v_j=1} u_j \mu_{ji}\right)} - 1 \right) = \sum_{j \neq i} (e^{u_j \mu_{ji}} - 1) + O(1/M),$$

984 which justifies relation (3.9) under assumption that the involved expectation terms  
 985 remain bounded when  $M \rightarrow \infty$ .

986 **5.2. Solutions to the RMF ansatz.** Solving the RMF *ansatz* for the relaxing-  
 987 neuron model with synaptic heterogeneity is more involved than for the counting-  
 988 neuron model. This is primarily due to the fact that in the presence of relaxation,  
 989 stochastic intensities have a continuous state space, which requires to consider MGFs  
 990 instead of PGFs. The defining property of MGFs is provided by the criterion of com-  
 991 plete monotonicity. To prove [Theorem 3.8](#), we first show that the RMF *ansatz* admits  
 992 a unique smooth solution ([Subsection 5.2.1](#)). Then, we show that this smooth solu-  
 993 tion is completely monotone ([Subsection 5.2.2](#)). Finally, we show that the condition  
 994 of normalization for smooth solutions reduces to the announced set of equations for  
 995 the mean neuronal intensities, which admits at least one solution ([Subsection 5.2.3](#)).

996 **5.2.1. Uniqueness of smooth solutions.** Just as for the counting-neuron  
 997 model, there is a unique smooth solution to the type of ODEs intervening in the RMF  
 998 *ansatz* for the relaxing-neuron model with synaptic heterogeneity. This is stated in  
 999 the following proposition:

1000 **PROPOSITION 5.2.** *Let  $f$  and  $g$  be real-valued functions in  $C^{n+1}(\mathbb{R})$  with  $n \geq 1$*   
 1001 *and  $\tau$  a positive real number such that  $f(-\tau) > 0$ , then the ODE*

$$1002 \quad (5.20) \quad \left(1 + \frac{u}{\tau}\right) L'(u) + f(u)L(u) - g(u) = 0,$$

1003 *admits a unique continuous solution on  $\mathbb{R}$ :*

$$1004 \quad (5.21) \quad L(u) = \int_{-\tau}^u e^{-\int_v^u \frac{f(w)}{1+w/\tau} dw} \frac{g(v)}{1+v/\tau} dv.$$

1005 *Moreover, this solution admits a derivative of order  $n$  in  $-\tau$ . In particular, we have*

$$1006 \quad (5.22) \quad L(-\tau) = g(-\tau)/f(-\tau) \quad \text{with} \quad L'(-\tau) = \frac{(g/f)'(-\tau)}{1 + (\tau f(-\tau))^{-1}}.$$

1007 *Proof. i) Uniqueness.* As  $f$  and  $g$  are continuous on  $\mathbb{R}$ , (3.12) admits continuously  
 1008 differentiable solutions on  $(-\infty, -\tau)$  and  $(-\tau, +\infty)$ . Solutions defined on  $(-\tau, +\infty)$   
 1009 have the generic integral expression

$$1010 \quad (5.23) \quad L(u) = L_0 e^{-\int_0^u \frac{f(v)}{1+v/\tau} dv} + \int_0^u e^{-\int_v^u \frac{f(w)}{1+w/\tau} dw} \frac{g(v)}{1+v/\tau} dv,$$

1011 where  $L_0$  denotes the arbitrary real value taken by  $L$  in zero. The analysis of the above  
 1012 expression shows that solutions on  $(-\tau, +\infty)$  generically have an infinite discontinuity  
 1013 when  $u \rightarrow -\tau^+$ . In fact, we evaluate by integration by parts that

$$1014 \quad (5.24) \quad \frac{1}{\tau} \int_0^u \frac{f(v)}{1+v/\tau} dv = \left[ f(v) \ln \left( 1 + \frac{v}{\tau} \right) \right]_0^u - \int_0^u f'(v) \ln \left( 1 + \frac{v}{\tau} \right) dv,$$

1015 where the integral in the right-hand term has a finite limit when  $u \rightarrow -\tau^+$ . Thus,  
 1016 the homogeneous part of  $L$  exhibits the asymptotic behavior

$$1017 \quad (5.25) \quad e^{-\int_0^u \frac{f(v)}{1+v/\tau} dv} \sim c \left( 1 + \frac{u}{\tau} \right)^{-a}, \quad u \rightarrow -\tau^+$$

1018 where we have set the constants

$$1019 \quad (5.26) \quad a = \tau f(-\tau) > 0 \quad \text{and} \quad c = -\tau \int_{-\tau}^0 f'(v) \ln \left( 1 + \frac{v}{\tau} \right) dv,$$

1020 thereby showing that  $L$  generically has an infinite discontinuity in  $-\tau$ . Factorizing  
 1021 the homogeneous part leads to considering  $L$  under the form

$$1022 \quad (5.27) \quad L(u) = e^{-\int_0^u \frac{f(v)}{1+v/\tau} dv} \left( L_0 + \int_0^u e^{-\int_v^0 \frac{f(w)}{1+w/\tau} dw} \frac{g(v)}{1+v/\tau} dv \right).$$

1023 For  $L$  to have a finite left-limit in  $-\tau$ , the term in parentheses in the above expression  
 1024 must vanish when  $u \rightarrow -\tau^+$ , which implies that one must choose

$$1025 \quad (5.28) \quad L_0 = \lim_{u \rightarrow -\tau^+} \int_u^0 e^{-\int_v^0 \frac{f(w)}{1+w/\tau} dw} \frac{g(v)}{1+v/\tau} dv.$$

1026 The above limit exists and is finite due to the asymptotic behavior of the integrand

$$1027 \quad (5.29) \quad e^{-\int_v^0 \frac{f(w)}{1+w/\tau} dw} \frac{g(v)}{1+v/\tau} \sim \frac{g(-\tau)}{c} \left( 1 + \frac{v}{\tau} \right)^{a-1},$$

1028 where the right-hand term is integrable ( $a > 0$ ). This shows that a continuous solution  
 1029 to (5.20) must take a unique value  $L_0$  in 0 and is therefore uniquely characterized on  
 1030  $(-\tau, +\infty)$ . Moreover, inserting the integral expression for  $L_0$  given by (5.27) into  
 1031 (5.23) yields the announced expression (5.21) for that unique solution. Repeating the  
 1032 above analysis on  $(-\infty, -\tau)$  rather than  $(-\tau, +\infty)$  would yield the same expression  
 1033 for the unique solution with a finite right-limit in  $-\tau$ , showing that there is at most  
 1034 one continuous solution to (5.23) on  $\mathbb{R}$ .

1035 *ii) Existence: continuity.* It is enough to show that the function  $L$  defined on  
 1036  $\mathbb{R} \setminus \{\tau\}$  by (5.21) is continuous in  $-\tau$ . In order to compute  $\lim_{u \rightarrow \tau} L(u)$ , we first use  
 1037 integration by part to obtain the asymptotic behavior of the exponent function in  
 1038 (5.21) when  $u \rightarrow -\tau$ :

$$1039 \quad \frac{1}{\tau} \int_u^v \frac{f(w)}{1+w/\tau} dw = \left[ f(w) \ln \left( \left| 1 + \frac{w}{\tau} \right| \right) \right]_u^v - \int_u^v f'(w) \ln \left( \left| 1 + \frac{w}{\tau} \right| \right) dw,$$

$$1040 \quad (5.30) \quad = f(-\tau) \ln \left( \left| \frac{\tau+v}{\tau+u} \right| \right) + o_{-\tau}(1), \quad |\tau+v| < |\tau+u|.$$

1041 Thus we have the equivalence

$$1042 \quad (5.31) \quad e^{-\int_v^u \frac{f(w)}{1+w/\tau} dw} \sim \left( \frac{\tau+v}{\tau+u} \right)^a, \quad 0 < \frac{\tau+v}{\tau+u} < 1, \quad u \rightarrow -\tau,$$

1043 which shows that the sought-after limit can be evaluated as:

$$1044 \quad (5.32) \quad \lim_{u \rightarrow -\tau} L(u) = \lim_{u \rightarrow -\tau} \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a \frac{g(v)}{1+v/\tau} dv.$$

1045 The leading term in the above integral can be further evaluated via integration by  
1046 part

$$1047 \quad \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a \frac{g(v)}{1+v/\tau} dv = \frac{\tau}{(\tau+u)^a} \int_0^{\tau+u} w^{a-1} g(w-\tau) dw,$$

$$1048 \quad (5.33) \quad = \frac{\tau}{(\tau+u)^a} \left( \left[ \frac{w^a}{a} g(w-\tau) \right]_0^{\tau+u} - \int_0^{\tau+u} \frac{w^a}{a} g'(w-\tau) dw \right),$$

1049 where the integral in the right-hand side is  $O_{-\tau}(\tau+u)$ . Taking the limit  $u \rightarrow -\tau$  in  
1050 the remaining term yields the announced value

$$1051 \quad (5.34) \quad L(-\tau) = \lim_{u \rightarrow -\tau} \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a \frac{g(v)}{1+v/\tau} dv = \lim_{u \rightarrow -\tau} \frac{\tau g(u)}{a} = \frac{g(-\tau)}{f(-\tau)},$$

1052 showing that  $L$  is continuous on  $\mathbb{R}$ .

1053 *iii) Differentiability.* Let us first evaluate  $L'(\tau)$  by Taylor expanding  $L(u)$  in  $-\tau$   
1054 to first order. First, by repeated integration by parts, we obtain

$$1055 \quad \frac{1}{\tau} \int_0^u \frac{f(v)}{1+v/\tau} dv = f(u) \ln \left( \left| 1 + \frac{u}{\tau} \right| \right) -$$

$$1056 \quad (5.35) \quad f'(u)(\tau+u) \left( \ln \left( \left| 1 + \frac{u}{\tau} \right| \right) - 1 \right) - \tau f'(0) + F(u),$$

1057 where the last term  $F(u)$  refers to the function continuously differentiable function

$$1058 \quad (5.36) \quad F(u) = \int_0^u f''(v)(\tau+v) \left( \ln \left( \left| 1 + \frac{v}{\tau} \right| \right) - 1 \right) dv.$$

1059 Noticing that  $F'(-\tau) = 0$ , we have  $F(v) - F(u) = o_{-\tau}(\tau+u)$  when  $|\tau+v| < |\tau+u|$ .  
1060 Moreover, Taylor expanding  $f$  and  $f'$  around  $-\tau$  yields

$$1061 \quad f(u) \ln \left( \left| 1 + \frac{u}{\tau} \right| \right) - f'(u)(\tau+u) \left( \ln \left( \left| 1 + \frac{u}{\tau} \right| \right) - 1 \right) =$$

$$1062 \quad (5.37) \quad f(-\tau) \ln \left( \left| 1 + \frac{u}{\tau} \right| \right) + f'(-\tau)(\tau+u) + o_{-\tau}(\tau+u).$$

1063 Thus, when  $u \rightarrow -\tau$ ,  $|\tau+v| < |\tau+u|$ , the first-order approximation to the exponent  
1064 function in (5.21) is

$$1065 \quad (5.38) \quad -\frac{1}{\tau} \int_v^u \frac{f(v)}{1+v/\tau} dv = f(-\tau) \ln \left( \left| \frac{\tau+v}{\tau+u} \right| \right) + f'(-\tau)(v-u) + o_{-\tau}(\tau+u).$$

1066 In turn, to first-order in  $\tau+u$ , we have the asymptotic behavior for  $L(u)$

$$1067 \quad (5.39) \quad L(u) = \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a \left( 1 + f'(-\tau)(v-u) \right) \frac{g(v)}{1+v/\tau} dv + o_{-\tau}(\tau+u).$$

1068 To write the above relation as an explicit linear approximation, we split the above  
 1069 expression in three terms that we evaluate separately:  $L(u) = A(u) + B(u) + C(u)$ .  
 1070 The linear approximation to the first term is obtained by repeated integration by part

$$\begin{aligned}
 1071 \quad (5.40) \quad A(u) &= \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a \frac{g(v)}{1+v/\tau} dv, \\
 1072 &= \frac{\tau g(u)}{a} - \frac{\tau g'(u)}{a(a+1)} (\tau+u) + o_{\tau+u}(\tau+u), \\
 1073 &= L(-\tau) + \frac{\tau g'(-\tau)}{a+1} (\tau+u) + o_{\tau+u}(\tau+u).
 \end{aligned}$$

1074 while the linear approximations to the remaining terms only requires one integration  
 1075 by part:

$$\begin{aligned}
 1076 \quad (5.41) \quad B(u) &= \tau f'(-\tau) \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a (\tau+u) \frac{g(v)}{1+v/\tau} dv, \\
 1077 &= \frac{\tau^2 f'(-\tau) g(-\tau)}{a} (\tau+u) + o_{\tau+u}(\tau+u).
 \end{aligned}$$

$$\begin{aligned}
 1079 \quad (5.42) \quad C(u) &= \tau f'(-\tau) \int_{-\tau}^u \left( \frac{\tau+v}{\tau+u} \right)^a (\tau+v) \frac{g(v)}{1+v/\tau} dv, \\
 1080 &= \frac{\tau^2 f'(-\tau) g(-\tau)}{a+1} (\tau+u) + o_{\tau+u}(\tau+u).
 \end{aligned}$$

1081 Remembering that  $a = \tau f(-\tau)$ , we find the announced limit behavior

$$\begin{aligned}
 1082 \quad (5.43) \quad \lim_{u \rightarrow -\tau} \frac{L(u) - L(-\tau)}{\tau+u} &= \frac{\tau g'(-\tau)}{a+1} - \tau^2 f'(-\tau) g(-\tau) \left( \frac{1}{a} - \frac{1}{a+1} \right), \\
 1083 &= \frac{a}{a+1} \left( \frac{g}{f} \right)' (-\tau).
 \end{aligned}$$

1084 Derivatives of higher order are obtained via similar, albeit intricate, calculations eval-  
 1085 uating the higher-order Taylor expansions of  $L(u)$  around  $-\tau$ . The maximum order  
 1086 for this expansion is determined by the number of times that integration by part can  
 1087 be performed in step (5.35) and step (5.40). The maximum order is therefore  $n-1$   
 1088 for functions  $f$  and  $g$  in  $C^{(n)}(\mathbb{R})$ , which implies that  $L$  has a derivative of order  $n-1$   
 1089 in  $-\tau$ .  $\square$

1090 *Remark 5.3.* Proposition 5.2 actually holds for equations of the form

$$1091 \quad (5.44) \quad h(u+\tau)L'(u) + f(u)L(u) - g(u) = 0,$$

1092 where  $h$  is continuously differentiable with a single root:  $h(0) = 0$ ,  $h'(0) > 0$ . Knowing  
 1093 continuous differentiability, the value

$$1094 \quad (5.45) \quad L'(-\tau)(u+\tau) = \frac{(g/f)'(-\tau)}{1+h'(0)/f(-\tau)},$$

1095 directly follows from linearizing (5.44) and from using  $L(-\tau) = g(-\tau)/f(-\tau)$ .



1096 **5.2.2. Complete monotonicity of the smooth solution.** The following lem-  
 1097 ma will be the key to prove the complete monotonicity of the smooth solutions to the  
 1098 RMF *ansatz*.

1099 LEMMA 5.4. *Let  $f$  and  $g$  be real-valued functions in  $C^2(\mathbb{R})$  such that  $f > 0$ ,  $g > 0$ ,*  
 1100 *and  $f' < 0$ ,  $g' > 0$  on an open interval  $I$  containing  $-\tau$ . Then, the unique continuous*  
 1101 *solution  $L$  to (5.20) is strictly increasing on  $I$ .*

1102 *Proof.* If  $g > 0$  and  $f > 0$ , expression (5.21) directly shows that  $L$  remains positive  
 1103 on  $\mathbb{R}$ . As  $L$  is solution to (5.44) and  $f > 0$  on  $I$ ,  $L$  is increasing on  $I$  if and only if  
 1104  $L \geq g/f$  on  $(-\infty, -\tau) \cap I$  and  $L(u) \leq g(u)/f(u)$  on  $(-\tau, \infty) \cap I$ . Let us show that  
 1105  $L$  is below the curve of  $g/f$  on  $(-\tau, \infty) \cap I$  by contradiction. First, observe that by  
 1106 Proposition 5.2, we know that the curve of  $L$  intersects the curve of  $g/f$  in  $-\tau$  with  
 1107 a slope  $L'(-\tau) < (g/f)'(-\tau)$ . In particular,  $L < g/f$  on the interval  $(-\tau, -\tau + \epsilon)$  for  
 1108 small enough  $\epsilon > 0$ . Suppose there is  $u$  in  $I$ ,  $u > -\tau + \epsilon$ , such that  $L(u) > g/f(u)$ ,  
 1109 then the set

$$1110 \quad (5.46) \quad V = \{v \in I \cap (-\tau + \epsilon, +\infty) \mid L(v) = g(v)/f(v)\}$$

1111 is non empty by continuity of  $L$  and  $g/f$ . Consider the first hitting time:  $v_0 = \inf V >$   
 1112  $-\tau$ . By definition,  $L$  remains below  $g/f$  on  $(-\tau, v_0)$  and we must have  $L'(v_0) = 0$ .  
 1113 However,  $f/g$  is a strictly increasing function when  $f > 0$ ,  $g > 0$ , and  $f' < 0$ ,  $g' > 0$ .  
 1114 Thus,  $(f/g)'(v_0) > 0 = L'(v_0)$  while  $(f/g)(v_0) = L(v_0)$ , which implies that  $f/g < L$   
 1115 in the left vicinity of  $v_0$ . This contradicts the definition of  $v_0$  as the first-hitting time.  
 1116 The same argument applies on  $(-\tau, \infty)$  to show that the curve of  $L$  above the curve  
 1117 of  $g/f$  on  $(-\infty, -\tau)$ .  $\square$

1118 We are now in a position to prove a result of monotonicity for derivatives of  
 1119 all orders via a simple recurrence argument, which is equivalent to the property of  
 1120 complete monotonicity.

1121 PROPOSITION 5.5. *Let  $f$  and  $g$  be real-valued functions in  $C^\infty(\mathbb{R})$  such that for*  
 1122 *all  $u < 0$ , we have  $f(u) > 0$ ,  $g(u) > 0$  and  $f^{(n)}(u) < 0$ ,  $g^{(n)}(u) > 0$  for all  $n$  in  $\mathbb{N}_*$ .*  
 1123 *Then, the unique continuous solution  $L$  to (5.20) is such that for all  $n$  in  $\mathbb{N}$  and for*  
 1124 *all  $u < 0$ , we have  $L^{(n)}(u) > 0$ .*

1125 *Proof.* *i)* The first step is to exhibit a system of first-order ODEs satisfied by the  
 1126  $(n+1)$ -th order derivatives  $L^{(n+1)}$ . Proposition 5.2 directly implies that the continuous  
 1127 solution  $L$  to Equation (5.20) is in  $C^\infty(\mathbb{R})$  on  $\mathbb{R}$  if  $f$  and  $g$  are in  $C^\infty(\mathbb{R})$ . Repeated  
 1128 differentiation of (5.20) on  $\mathbb{R} \setminus \{-\tau\}$  shows that for all  $n$  in  $\mathbb{N}$ , the functions  $L^{(n+1)}$   
 1129 satisfy

$$1130 \quad (5.47) \quad \left(1 + \frac{u}{\tau}\right) L^{(n+1)}(u) + f_n(u)L^{(n)}(u) - g_n(u) = 0,$$

1131 where we have  $f_n = n/\tau + f$  and where the function  $g_n$  is defined by recurrence as

$$1132 \quad (5.48) \quad g_n(u) = g'_{n-1}(u) - f'(u)L^{(n-1)}(u), \quad \text{with } g_0 = g(u).$$

1133 Proceeding inductively, we obtain an explicit expression for  $g_n$ :

$$1134 \quad (5.49) \quad g_n(u) = g^{(n)}(u) - \sum_{k=0}^{n-1} \frac{d^k}{du^k} \left( f'(u)L^{(n-1-k)}(u) \right),$$

1135 which can be further simplified via the Leibniz formula and the hockey-stick identity

$$\begin{aligned}
 1136 \quad (5.50) \quad g_n(u) &= g^{(n)}(u) - \sum_{k=0}^{n-1} \sum_{l=0}^k \binom{k}{l} f^{(l+1)}(u) L^{(n-1-l)}(u), \\
 1137 \quad (5.51) \quad &= g^{(n)}(u) - \sum_{l=0}^{n-1} \binom{n}{l+1} f^{(l+1)}(u) L^{(n-1-l)}(u).
 \end{aligned}$$

1138 *ii)* The proof then proceeds by recurrence on the order of the derivative. We know  
 1139 that the unique continuous solution to (5.20) is a positive function:  $L > 0$ . Suppose  
 1140 that  $L^{(k)} > 0$ , for  $1 \leq k \leq n$ , i.e., that the functions  $L^{(k)}$ ,  $0 \leq k \leq n-1$ , are positive  
 1141 increasing functions on  $\mathbb{R}_-$ . Formula (5.50) shows that  $g_n$  is also positive increasing:  
 1142  $g_n > 0$  and  $g'_n > 0$ . Then, observing that  $f_n$  and  $g_n$  in (5.44) satisfy the hypotheses of  
 1143 Lemma 5.4 with  $I = (-\infty, 0)$ , we conclude that  $L^{(n)}$  is positive increasing on  $(-\infty, 0)$ ,  
 1144 i.e.  $L^{(n+1)} > 0$ . By recurrence, we deduce that derivatives of all order are positive:  
 1145  $L^{(n)} > 0$  on  $(-\infty, 0)$  for all  $n \in \mathbb{N}$ .  $\square$

1146 **5.2.3. Existence of a solution to the RMF ansatz.** The proof of Theo-  
 1147 rem 3.8 mirrors the argument of the proof of Theorem 3.4, except that one has to  
 1148 check that *i)* the smooth solutions of the RMF ansatz are indeed MGFs and *ii)* that  
 1149 the self-consistency equations for the mean neuronal intensities admit at least one  
 1150 solution.

1151 *Proof of Theorem 3.8. i) Necessary conditions on the mean intensities.* Given  
 1152 positive mean intensities  $\beta_j > 0$ ,  $1 \leq j \leq K$ , each equation of the system (3.12) can  
 1153 be written under the same form as (5.21) by introducing the functions

$$1154 \quad (5.52) \quad f_i(u) = -\frac{ub_i}{\tau_i} + \sum_{j \neq i} (1 - e^{u\mu_{ij}}) \beta_j \quad \text{and} \quad g_i(u) = \beta_i e^{ur_i},$$

1155 which belong to  $C^\infty(\mathbb{R})$  with  $f_i(-\tau_i) > 0$ . Thus, by Proposition 5.2, each equation of  
 1156 the system (3.12) admits the unique continuous solution on  $\mathbb{R}$

$$1157 \quad (5.53) \quad L_i(u) = \int_{-\tau_i}^u e^{-\int_v^u \frac{f_i(w)}{1+\tau_i/w} dw} \frac{g_i(v)}{1+v/\tau_i} dv, \quad 1 \leq j \leq K,$$

1158 which also belong to  $C^\infty(\mathbb{R})$ . Moreover, the functions  $f_i$  and  $g_i$  are such that for all  
 1159  $u < 0$ ,  $f_i(u) > 0$ ,  $g_i(u) > 0$ ,  $f_i^{(n)}(u) < 0$  and  $g_i^{(n)}(u) > 0$  if  $\beta_j > 0$  for  $1 \leq j \leq K$ .  
 1160 Thus, by Proposition 5.5, we deduce that the functions  $L_i$ ,  $1 \leq j \leq K$ , have strictly  
 1161 positive derivative at all order in  $(-\infty, 0)$ . Together, the above properties state that  
 1162 the functions defined by  $u \mapsto L_i(-u)$  are completely monotone function on  $(0, \infty)$   
 1163 [27]. By Bernstein's theorem on completely monotone functions,  $u \mapsto L_i(-u)$  is the  
 1164 Laplace transform of a positive measure  $m_i$  defined on the Borel sets of  $\mathbb{R}_+$ , that is:

$$1165 \quad (5.54) \quad L_i(-u) = \int_0^\infty e^{-ut} dm_i(t).$$

1166 In particular, the functions  $L_i$  are MGFs if and only if the measures  $m_i$  are probability  
 1167 measure. This is equivalent to imposing that  $L_i(0) = 1$ ,  $1 \leq i \leq K$ , which gives the  
 1168 announced system of equations (3.47) for the mean intensities  $\beta_j$ . Operating the

1169 change of variables  $y = \tau_i \ln(1 + v/\tau_i)$  and  $x = \tau_i \ln(1 + w/\tau_i)$  yields the integral  
 1170 expression

1171  $L_i(u) = \beta_i \times$

$$1172 \quad (5.55) \quad \int_{-\infty}^u \exp \left( \int_y^u b_i \left( e^{\frac{x}{\tau_i}} - 1 \right) + \sum_{j \neq i} \left( e^{\tau_i \mu_{ij} \left( e^{\frac{x}{\tau_i}} - 1 \right)} - 1 \right) \beta_j dx \right) e^{\tau_i r_i \left( e^{\frac{y}{\tau_i}} - 1 \right)} dy,$$

1173 which reduces to (3.44) after evaluating the integral exponent, therefore justifying the  
 1174 announced system of equations (3.47) for the mean intensities  $\beta_j$ .

1175 *ii) Existence of mean intensities solutions.* In order to show the existence of  
 1176 solutions to the system of equations (3.47), let us consider the map  $\mathbf{F} : \mathbb{R}_+^K \rightarrow \mathbb{R}_+^K$   
 1177 whose components are defined by

$$1178 \quad (5.56) \quad F_i(\boldsymbol{\beta}) = \left( \int_{-\infty}^0 \exp \left( \left[ h_i(x) - \sum_{j \neq i} \beta_j h_{ij}(x) \right]_v^0 + l_i(v) \right) dv \right)^{-1}, \quad 1 \leq i \leq K.$$

1179 Given  $\boldsymbol{\beta}_0$  in the positive orthant, iterating the map  $\mathbf{F}$  specifies a sequence  $\{\boldsymbol{\beta}_n\}_{n \in \mathbb{N}}$ ,  
 1180  $\boldsymbol{\beta}_n = \mathbf{F}^n(\boldsymbol{\beta}_0)$ , whose finite accumulation points are solutions to (3.47). To establish  
 1181 that such accumulation points exist, it is enough to show that the positive sequence  
 1182  $\{\boldsymbol{\beta}_n\}_{n \in \mathbb{N}}$  is bounded. Given  $\boldsymbol{\beta}_0$  in the positive orthant, we show the boundedness of  
 1183  $\{\boldsymbol{\beta}_n\}_{n \in \mathbb{N}}$  by exhibiting a dominating convergent sequence. The first step is to observe  
 1184 that for  $t \leq 0$ , we have:

$$1185 \quad (5.57) \quad h_i(0) - h_i(x) + l_i(x) = \tau_i(r_i - b_i) \left( e^{x/\tau_i} - 1 \right) + b_i x \geq \max(b_i, r_i)x,$$

1186 and consequently, we have

$$1187 \quad (5.58) \quad F_i(\boldsymbol{\beta}) \leq \left( \int_{-\infty}^0 \exp \left( \max(b_i, r_i)v - \sum_{j \neq i} \beta_j \left[ h_{ij}(x) \right]_v^0 \right) dv \right)^{-1} \stackrel{\text{def}}{=} \tilde{F}_i(\boldsymbol{\beta}).$$

1188 Because of the convexity of the exponential function, the newly introduced function  
 1189  $\tilde{F}_i$  turns out to be an increasing function of the relaxation time  $\tau_i$ , so that we have

$$1190 \quad (5.59) \quad F_i(\boldsymbol{\beta}) \leq \lim_{\tau_i \rightarrow \infty} \tilde{F}_i(\boldsymbol{\beta}) = \left( \int_{-\infty}^0 \exp \left( r_i t + \sum_{j \neq i} \beta_j \left( \frac{1 - e^{t \mu_{ij}}}{\mu_{ij}} + t \right) \right) dt \right)^{-1}.$$

1191 Observing that  $\lim_{\tau_i \rightarrow \infty} \tilde{F}_i(\boldsymbol{\beta})$  is also an increasing function of the parameters  $\mu_{ij}$   
 1192 and  $r_i$ , we further have

$$1193 \quad (5.60) \quad F_i(\boldsymbol{\beta}) \leq \left( \int_{-\infty}^0 \exp \left( rt + \left( \frac{1 - e^{t\mu}}{\mu} + t \right) \sum_{j \neq i} \beta_j \right) dt \right)^{-1} \stackrel{\text{def}}{=} G_i(\boldsymbol{\beta}),$$

1194 where  $r = \max_i r_i$  and  $\mu = \max_{i,j} \mu_{ij}$ . As expected, evaluating the integral in the  
 1195 above expression for  $\boldsymbol{\beta} = \beta \mathbf{1}$  yields the equation associated to the counting-neuron  
 1196 model with interaction weight  $\mu$  and base intensity equal to the reset value  $r$ :

$$1197 \quad (5.61) \quad G_i(\beta \mathbf{1}) = \frac{\mu c^{c+x} e^{-c}}{\gamma(c+x, c)} \stackrel{\text{def}}{=} g(\beta), \quad \text{with } c = \frac{(K-1)\beta}{\mu} \quad \text{and } x = \frac{r}{\mu}.$$

1198 Given  $\beta_0$  in the positive orthant, posit  $\beta'_0 = (\max_i \beta_{0,i})\mathbf{1}$  and consider the two se-  
 1199 quences  $\{\beta_n\}_{n \in \mathbb{N}}$  and  $\{\beta'_n\}_{n \in \mathbb{N}}$  obtained by iterating the maps  $\mathbf{F}$  and  $\mathbf{G}$  on  $\beta_0$  and  
 1200  $\beta'_0$ , respectively:  $\beta_n = \mathbf{F}^n(\beta_0)$  and  $\beta'_n = \mathbf{G}^n(\beta'_0)$ . If  $\beta_n \leq \beta'_n$ , then  $\beta_{n+1} = \mathbf{F}(\beta_n) \leq$   
 1201  $\mathbf{F}(\beta'_n) \leq \mathbf{G}(\beta'_n) = \beta'_{n+1}$ , where we have used the fact that for all  $1 \leq i \leq K$ ,  $F_i$  is  
 1202 increasing with respect to  $\beta_j$ ,  $1 \leq j \leq K$ :

$$1203 \quad (5.62) \quad \partial_{\beta_j} F_i(\beta) = -\frac{\int_{-\infty}^0 h_{ij}(t) e^{(h_i(t) + \sum_{j \neq i} \beta_j h_{ij}(t))} dt}{F_i(\beta)^2} \geq 0.$$

1204 Thus, as  $\beta_0 \leq \beta'_0$  by construction, the sequence  $\{\beta'_n\}_{n \in \mathbb{N}}$  dominates  $\{\beta_n\}_{n \in \mathbb{N}}$  with re-  
 1205 spect to the product order in  $\mathbb{R}_K$ . It remains to show to  $\{\beta'_n\}_{n \in \mathbb{N}}$  is convergent, which  
 1206 is equivalent to show that the one dimensional sequence  $\{\beta'_n\}_{n \in \mathbb{N}}$ ,  $\beta'_n = g^n(\max_i \beta_{0,i})$ ,  
 1207 is convergent. To justify this point, it is enough to check that the sequence  $\{\beta'_n\}_{n \in \mathbb{N}}$   
 1208 is bounded, as Lemma 3.5 shows that there is a unique fixed point solution to  
 1209  $\beta = g(\beta) = \mu c^a e^{-c} / \gamma(a, c)$ . Introducing the rescaled sequence  $\{c_n\}_{n \in \mathbb{N}}$  defined by  
 1210  $c_n = (K-1)\beta'_n / \mu$ , notice that  $c_{n+1} = h(c_n)$  with

$$1211 \quad (5.63) \quad h(c) = (K-1) \frac{c^{c+x} e^{-c}}{\gamma(c+x, c)}.$$

1212 From the power expansion of the incomplete gamma function, we have

$$1213 \quad h(c) = (K-1) \left( \sum_{n \geq 0} \frac{c^n}{(x+c)(x+c+1) \dots (x+c+n)} \right)^{-1}$$

$$1214 \quad (5.64) \quad \leq (K-1) \left( \sum_{n=0}^{K-1} \frac{c^n}{(x+c)(x+c+1) \dots (x+c+n)} \right)^{-1} = \frac{K-1}{K} c + o_\infty(c).$$

1215 showing that  $h(c) < c$  for large enough  $c$ . This implies that  $\{c_n\}_{n \in \mathbb{N}}$  is a bounded  
 1216 sequence, and so is  $\{\beta'_n\}_{n \in \mathbb{N}}$ .  $\square$

1217 **6. Future directions.** Our results were obtained and discussed for purely exci-  
 1218 tatory LGL networks and limited to first-order RMF *ansatz*. We would like to stress  
 1219 that, in principle, our approach to reduce RMF *ansätze* to a set of self-consistency  
 1220 equations—founded on imposing the condition of analyticity on the solutions to the  
 1221 *ansätze*—can be generalized to models including inhibition and higher-order statistics.

1222 In the context of second-order RMF, the RCP can be applied to the joint MGF of  
 1223 pairs of neurons rather than single neurons. Our replica framework can be extended  
 1224 to simplify the representation of the point processes that feed this pair through some  
 1225 appropriate extension of the Poisson Hypothesis. The interactions between the two  
 1226 neurons of the pair are however described in an exact way. An important complication  
 1227 of our replica approach for higher order is that the RMF *ansatz* consists in a system  
 1228 of PDEs rather than a system of ODEs. However, the PDEs associated with the  
 1229 RCP for second-order RMF model can be solved using singularity-analysis techniques  
 1230 generalizing those described in this work. This line of thought is essential to represent,  
 1231 e.g., the wave phenomena present in cyclic networks, which limits the applicability  
 1232 of first-order RMF networks. Second-order RMF networks are expected to bring  
 1233 essential new features absent from order one. They are most probably the least  
 1234 complex networks within the RMF class allowing one to capture correlation effects.  
 1235 They also seem to provide the least complex networks that are not fundamentally  
 1236 time irreversible, i.e., with a positive production of entropy.

1237 Another important extension is to account for networks supporting both exci-  
 1238 tatory and inhibitory interactions within our RMF framework. Including inhibitory  
 1239 interactions within a point-process framework requires to consider nonlinear mod-  
 1240 els of synaptic integration, whereby stochastic intensities can remain non-negative in  
 1241 spite of inhibitory inputs. There are several possible nonlinear models which are bio-  
 1242 physically relevant, each yielding distinct functional characterizations of their RMF  
 1243 stationary state. Considering these nonlinear RMF networks in toy models shows that  
 1244 singularity-analysis techniques are still applicable to networks with mixed excitation  
 1245 and inhibition. However, the presence of inhibition fundamentally alters the nature of  
 1246 the singularity featuring in the non-physical solutions to the RMF *ansatz*. Generaliz-  
 1247 ing our analysis to singularities that are more involved than infinite discontinuities is  
 1248 the key challenge to include inhibition within our framework. Importantly, we have  
 1249 numerical evidence that networks with inhibition have RMF versions that admit sev-  
 1250 eral stable solutions. We intend to utilize these multistable RMF networks to probe  
 1251 the metastable behavior of the finite-size networks that share the same neural basic  
 1252 structure.

1253 The above computational questions will be explored in companion papers. A  
 1254 more fundamental question remains to prove the propagation of chaos in finite-replica  
 1255 models, which is supported by simulations and is the central conjecture of this work.

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